MANONMANIAM SUNDARANAR UNIVERSITY

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DIRECTORATE OF DISTANCE AND CONTINUING EDUCATION

III B.Sc. MATHEMATICS

SEMESTER V

REAL ANALYSIS

Sub. Code: JMMA52

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REAL ANALYSIS (JMMA52)

UNIT	DETAILS
I	Metric spaces : Definition and Examples – Bounded sets – Open ball – open sets – Subspaces – Interior of a set.
II	Closed sets – Closure – Limit point – Dense set. Complete metric space : Completeness – Cantor's Intersection Theorem – Baire's Category theorem.
III	Continuity : Continuity – Homeomorphism – Uniform Continuity – Discontinuous functions on R.
IV	Connectedness : Definition and Examples – Connected subsets of R – Connectedness and continuity – Contraction mapping theorem.
V	Compactness : Compact metric spaces – Compact Subsets of R – Equivalent characterizations for compactness – Compactness and Continuity.

Recommended Text
S. Arumugam and A. Thangapandi Issac, Modern Analysis, New Gamma Publishing House, Palayamkottai, 2015

UNIT I

METRIC SPACES

Definition : A **metric space** is a non-empty set M together with a function $d: M \times M \to R$ satisfying the following conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in M$

(ii) $d(x, y) = 0$ iff $x = y$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$ (triangle inequality)

d is called a metric or distance function and d(x,y) is called the distance between x and y.

Note : The metric space M with the metric d is denoted by (M, d) or simply by M when the underlying metric d is clear from the context.

Example 1 : In R we defind $d(x, y) = |x - y|$. Then d is a metric on R. This is called the usual metric on R.

Proof : Clearly $d(x, y) = |x - y| \geq 0$.

Also, $d(x, y) = 0 \Leftrightarrow |x - y| = 0 \Leftrightarrow x = y$.

$$d(x, y) = |x - y| = |y - x| = d(y, x).$$

Now, let $x, y, z \in R$.

Then, $d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z)$.

$$\therefore d(x, z) \leq d(x, y) + d(y, z).$$

Hence, d is a metric on M.

Note : Whenever we consider R as a metric space the underlying metric is taken to b the usual metric unless otherwise stated.

Example 2 : In C, we define $d(z, w) = |z - w|$. Then d is a metric on C. This is called the usual metric on C.

Note : If the complex number $z = x + iy$ is identified with the point (x,y) of the two dimensional Euclidean plane then the above distance formula takes the form $d(z, w) = \sqrt{(x - u)^2 + (y - v)^2}$ where $z = x + iy$ and $w = u + iv$. This is nothing but the usual distance between the points (x, y) and (u, v) in the plane.

Example 3 : On any non-empty set M we define d as follows:

$$d(x,y) = \begin{cases} 0 \; if \; x = y \\ 1 \; if \; x \neq y \end{cases}$$

Then d is a metric on M. This is called the discrete metric on M.

Proof : Clearly, $d(x,y) \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$.

Also, $d(x,y) = d(y,x) = \begin{cases} 0 \; if \; x = y \\ 1 \; if \; x \neq y \end{cases}$

$\therefore d(x,y) = d(y,x)$ for all $x, y \in M$.

Now let $x, y, z \in M$

Case (i) $x = z$

Then $d(x,z) = 0$.

Also, $d(x,y) + d(y,z) \geq 0$.

$$\therefore d(x,z) \leq d(x,y) + d(y,z).$$

Case (ii) $x \neq z$

Then $d(x,z) = 1$.

Also, since x, z are distinct, y cannot be equal to both x and z.

Hence, either $y \neq x$ or $y \neq z$.

$$\therefore d(x,y) + d(y,z) \geq 1.$$

$$\therefore d(x,z) \leq d(x,y) + d(y,z).$$

Thus $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in M$.

Hence, d is a metric on M.

Example 4 : In R^n we define $d(x,y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\;1/2}$ where $x = (x_1, x_2, \ldots., x_n)$ and $y = (y_1, y_2, \ldots \ldots . y_n)$. Then d is a metric on R^n. This is called the usual metric on R^n.

Proof : $d(x,y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\;1/2} \geq 0$.

$$d(x,y) = 0 \Leftrightarrow \left[\sum_{i=1}^{n}(x_i - y_i)^2\right]^{1/2} = 0$$

$\Leftrightarrow (x_i - y_i)^2 = 0$ for all i=1,2,......n.

$$\Leftrightarrow x_i = y_i \; for \; all \; i = 1,2, \ldots . . n.$$

$$\Leftrightarrow (x_1, x_2, \ldots., x_n) = (y_1, y_2, \ldots \ldots . y_n)$$

$$\Leftrightarrow x = y.$$

Also, $d(x,y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{\;1/2} = [\sum_{i=1}^{n}(y_i - x_i)^2]^{\;1/2} = d(y,x)$.

To prove the triangle inequality, take $a_i = x_i - y_i, b_i = y_i - z_i \; and \; p = 2$ in Minkowski's inequality we get, $[\sum_{i=1}^{n}(x_i - z_i)^2]^{\frac{1}{2}} \leq [\sum_{i=1}^{n}(x_i - y_i)^2]^{\;\frac{1}{2}} + [\sum_{i=1}^{n}(y_i - x_i)^2]^{\;\frac{1}{2}}$

$$i.e., d(x,z) \leq d(x,y) + d(y,z).$$

$\therefore$ d is a metric on R^n.

Note : R^n with usual metric is called the n-dimensional Euclidean space.

Example 5: Consider R^n. Let $p > 1$. we define $d(x,y) = [\sum_{i=1}^{n}(x_i - y_i)^p]^{1/p}$ where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots \ldots y_n)$. Then d is a metric on R^n.

The proof is similar to that of example 4.

Example 6 : Let $x, y \in R^2$. Then $x = (x_1, x_2)$ and $y = (y_1, y_2)$ where $x_1, x_2, y_1, y_2 \in R$. We define $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$. Then d is a metric on R^2.

Proof : $d(x,y) = |x_1 - y_1| + |x_2 - y_2| \geq 0$.

$$d(x,y) = 0 \Leftrightarrow |x_1 - y_1| + |x_2 - y_2| = 0$$

$$\Leftrightarrow |x_1 - y_1| = 0 \text{ and } |x_2 - y_2| = 0$$

$$\Leftrightarrow x_1 = y_1 \text{ and } x_2 = y_2$$

$$\Leftrightarrow (x_1, x_2) = (y_1, y_2)$$

$$\Leftrightarrow x = y.$$

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

$$= |y_1 - x_1| + |y_2 - x_2|$$

$$= d(y,x).$$

Now, let $x, y, z \in R^2$.

$$d(x,z) = |x_1 - z_1| + |x_2 - z_2|$$

$$= |x_1 - y_1 + y_1 - z_1| + |x_2 - y_2 + y_2 - z_2|$$

$$\leq \{|x_1 - y_1| + |y_1 - z_1|\} + \{|x_2 - y_2| + |y_2 - z_2|\}$$

$$= \{|x_1 - y_1| + |x_2 - y_2|\} + \{|y_1 - z_1| + |y_2 - z_2|\}$$

$$= d(x,y) + d(y,z).$$

$$\therefore d(x,z) \leq d(x,y) + d(y,z).$$

Hence d is a metric on R^n.

Example 8: Let $c_1, c_2, \ldots \ldots \ldots, c_n$ be given fixed positive real numbers. Let $x, y \in R^n$ where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots \ldots y_n)$. We define $d(x,y) = \sum_{i=1}^{n} c_i |x_i - y_i|$. Then d is a metric on R^n.

Note : A non-empty set M can be provided with different metrics. For example, R^n has been provided with five different metrics as seen from examples 4 to 8.

Example 9 : Let $p \geq 1$. Let l_p denote the set of all sequences (x_n) such that $\sum_1^\infty |x_n|^p$ is convergent. Define $d(x,y) = [\sum_{n=1}^\infty |x_n - y_n|^p]^{1/p}$ where $x = (x_n)$ and $y = (y_n)$. Then d is a metric on l_p.

Proof : Let $a, b \in l_p$.

First we prove that $d(a,b)$ is a real number.

By Minkowski's inequality we have,

$[\sum_{i=1}^n (a_i + b_i)^p]^{\frac{1}{p}} \leq [\sum_{i=1}^n |a_i|^p]^{\frac{1}{p}} + [\sum_{i=1}^n |b_i|^p]^{\frac{1}{p}}$ .... (1)

Since $a, b \in l_p$ the right hand side of (1) has a finite limit as $n \to \infty$.

$\therefore [\sum_{i=1}^n (a_i + b_i)^p]^{\frac{1}{p}}$ is aconvergent series.

Similarly we can prove that $[\sum_{i=1}^n (a_i - b_i)^p]^{\frac{1}{p}}$ is also convergent series and hence d(a,b) is a real number.

Now, taking limit as $n \to \infty$ in (1) we get

$[\sum_{i=1}^\infty (a_i + b_i)^p]^{\frac{1}{p}} \leq [\sum_{i=1}^\infty |a_i|^p]^{\frac{1}{p}} + [\sum_{i=1}^\infty |b_i|^p]^{\frac{1}{p}}$ .... (2)

Obviously $d(x,y) \geq 0$.

$$d(x,y) = 0 \ iff \ x = y$$

$$d(x,y) = d(y,x).$$

Now, let $x, y, z \in l_p$.

Taking $a_i = x_i - y_i \ and \ b_i = y_i - z_i$ in (2) we get

$$\left[\sum_{i=1}^\infty (x_i - y_i + y_i - z_i)^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^\infty |x_i - y_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^\infty |y_i - z_i|^p\right]^{\frac{1}{p}}$$

$$\left[\sum_{i=1}^\infty (x_i - z_i)^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^\infty |x_i - y_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^\infty |y_i - z_i|^p\right]^{\frac{1}{p}}$$

$$\therefore d(x,z) \leq d(x,y) + d(y,z).$$

Hence, d is a metric on l_p.

Note : In particular, l_2 is a metric space with the metric defined by $d(x,y) = [\sum_{n=1}^\infty |x_n - y_n|^2]^{1/2}$.

Example 10 : Let M be the set of all bounded real valued functions defined on a non-empty set E. Define $d(f,g) = \sup\{|f(x) - g(x)|/x \in E\}$. Then d is a metric on M.

Proof : $d(f,g) = sup\{|f(x) - g(x)|: x \in E\} \geq 0$.

Also, $d(f,g) = 0 \Leftrightarrow sup\{|f(x) - g(x)|: x \in E\} = 0$

$$\Leftrightarrow |f(x) - g(x)| = 0 \; for \; all \; x \in E$$

$$\Leftrightarrow f(x) = g(x) \; for \; all \; x \in E$$

$$\Leftrightarrow f = g.$$

Also, $d(f,g) = sup\{|f(x) - g(x)|\}$

$$= sup\{|g(x) - f(x)|\}$$

$$= d(g,f).$$

Now, let $f, g, h \in M$.

We have, $|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)|$

$$sup|f(x) - h(x)| \leq sup|f(x) - g(x)| + sup|g(x) - h(x)|$$

$$\therefore d(f,h) \leq d(f,g) + d(g,h).$$

Hence, d is a metric on M.

Example 11 : Let M be the set of all sequences in R. Let $x, y \in M$ and let $x = (x_n)$ and $y = (y_n)$. Define $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$. Then d is a metric on M.

Proof : Let $x, y \in M$. First we prove that d(x,y) is a real number ≥ 0.

We have $\frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)} \leq \frac{1}{2^n} \; for \; all \; n.$

Also, $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series.

$\therefore \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$ is a convergent series. [By Comparison test]

$\therefore d(x,y)$ is a real number and $d(x,y) \geq 0$.

$$d(x,y) = 0 \Leftrightarrow \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1 + |x_n - y_n|)} = 0$$

$$\Leftrightarrow |x_n - y_n| = 0 \; for \; all \; n$$

$$\Leftrightarrow x_n = y_n \; for \; all \; n$$

$$\Leftrightarrow x = y.$$

Also, $d(x,y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n(1+|x_n - y_n|)}$

$$= \sum_{n=1}^{\infty} \frac{|y_n - x_n|}{2^n(1 + |y_n - x_n|)}$$

$$= d(y,x).$$

Now, let $x, y, z \in M$. Then

$$\frac{|x_n - z_n|}{(1 + |x_n - z_n|)} = 1 - \frac{1}{(1 + |x_n - z_n|)}$$

$$\leq 1 - \frac{1}{(1 + |x_n - y_n| + |y_n - z_n|)}$$

$$= \frac{1 + |x_n - y_n| + |y_n - z_n| - 1}{(1 + |x_n - y_n| + |y_n - z_n|)}$$

$$= \frac{|x_n - y_n| + |y_n - z_n|}{(1 + |x_n - y_n| + |y_n - z_n|)}$$

$$= \frac{|x_n - y_n|}{(1 + |x_n - y_n| + |y_n - z_n|)} + \frac{|y_n - z_n|}{(1 + |x_n - y_n| + |y_n - z_n|)}$$

$$\leq \frac{|x_n - y_n|}{(1 + |x_n - y_n|)} + \frac{|y_n - z_n|}{(1 + |y_n - z_n|)}$$

Multiplying both sides of this inequality by $\frac{1}{2^n}$ and taking the limit from $n = 1\ to\ \infty$ we get $\sum_{n=1}^{\infty} \frac{|x_n - z_n|}{2^n(1+|x_n - z_n|)} \leq \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{(1+|x_n - y_n|)} + \sum_{n=1}^{\infty} \frac{|y_n - z_n|}{(1+|y_n - z_n|)}$.

$$\therefore d(x,z) \leq d(x,y) + d(y,z).$$

Hence, d is a metric on M.

Example 12 : Let l^{∞} denote the set of all bounded sequence of real numbers. let $x = (x_n)$ and $y = (y_n) \in l^{\infty}$. Define d on l^{∞} as $d(x,y) = lub|x_n - y_n|$. Then d is a metric on l^{∞}.

Proof : $d(x,y) = lub|x_n - y_n| \geq 0$.

$$d(x,y) = 0 \Leftrightarrow lub|x_n - y_n| = 0$$

$$\Leftrightarrow |x_n - y_n| = 0\ for\ 1 \leq n < \infty$$

$$\Leftrightarrow x_n = y_n\ for\ 1 \leq n < \infty$$

$$\Leftrightarrow (x_n) = (y_n) \Leftrightarrow x = y.$$

$$d(x,y) = lub|x_n - y_n|$$

$$= lub|y_n - x_n|$$

$$= d(y,x).$$

Now, let $z = (z_n)$.

Now, $|x_n - z_n| \leq |x_n - y_n| + |y_n - z_n|$

$$\leq lub|x_n - y_n| + lub|y_n - z_n|$$

$$= d(x,y) + d(y,z).$$

$$\therefore lub|x_n - z_n| \leq d(x,y) + d(y,z)$$

$$\therefore d(x,z) \leq d(x,y) + d(y,z).$$

Hence, d is a metric on l^{∞}.

Solved Problems :

Problem 1 : Let d_1 and d_2 be two metrics on M. Define $d(x,y) = d_1(x,y) + d_2(x.y)$. Prove that d is a metric on M.

Solution : Since, d_1 and d_2 are two metrics on M, we have

$d_1(x,y) \geq 0$ for all $x, y \in M$

$d_1(x,y) = 0$ iff $x = y$

$d_1(x,y) = d_1(y,x)$ for all $x, y \in M$

$d_1(x,z) \leq d_1(x,y) + d_1(y,z)$ for all $x, y, z \in M$

and

$d_2(x,y) \geq 0$ for all $x, y \in M$

$d_2(x,y) = 0$ iff $x = y$

$d_2(x,y) = d_2(y,x)$ for all $x, y \in M$

$d_2(x,z) \leq d_2(x,y) + d_2(y,z)$ for all $x, y, z \in M$

$d(x,y) = d_1(x,y) + d_2(x,y) \geq 0$

$$d(x,y) = 0 \Leftrightarrow d_1(x,y) + d_2(x.y) = 0$$

$$\Leftrightarrow d_1(x,y) = 0 \; and \; d_2(x.y) = 0$$

$$\Leftrightarrow x = y.$$

$$d(x,y) = d_1(x,y) + d_2(x.y)$$

$$= d_1(y,x) + d_2(y,x)$$

$$= d(y,x).$$

Now, let $x, y, z \in M$.

$$d_1(x,z) \leq d_1(x,y) + d_1(y,z) \& \; d_2(x,z) \leq d_2(x,y) + d_2(y,z)$$

$$Adding \; these \; two \; we \; get$$

$$d_1(x,z) + d_2(x,z) \leq \big(d_1(x,y) + \; d_2(x,y)\big) + (d_1(y,z) + d_2(y,z))$$

$$i.e., d(x,z) \leq d(x,y) + d(y,z).$$

$\therefore d$ is a metric on M.

Problem 2 : Determine whether $d(x,y)$ defined on R by $d(x,y) = (x-y)^2$ is a metric or not.

Solution : Let $x, y \in R$.

$d(x,y) = (x-y)^2 \geq 0.$

$$d(x,y) = 0 \Leftrightarrow (x-y)^2 = 0$$

$$\Leftrightarrow x = y.$$

$$d(x,y) = (x-y)^2 = (y-x)^2 = d(y,x).$$

But triangle inequality does not hold.

Take $x = -5, y = -4\ and\ z = 4$.

Then $d(x,y) = (-5+4)^2 = 1$

$$d(y,z) = (-4-4)^2 = 64$$

$$d(x,z) = (4+5)^2 = 81$$

Here, $d(x,z) > d(x,y) + d(y,z)$.

Hence triangle inequality does not hold.

$\therefore$ d is not a metric on R.

Problem 3 : If d is a metric on M, is d^2 is a metric on M?

Solution : Consider $d(x,y)$ defined on R by $d(x,y) = |x-y|$.

From Example 1, we have d is a metric on R.

But, $d^2(x,y) = |x-y|^2 = (x-y)^2$.

But d^2 is not a metric. [From Problem 2].

Problem 4 : If d is a metric on M, prove that $\sqrt{d}$ is a metric on M.

Solution : Let $x, y, z \in M$.

Since, $d(x,y) \geq 0$, we have $\sqrt{d(x,y)} \geq 0$.

Also, $\sqrt{d(x,y)} = \sqrt{d(y,x)}$

Now, $d(x,z) \leq d(x,y) + d(y,z)$

$$\therefore \sqrt{d(x,z)} \leq \sqrt{d(x,y) + d(y,z)}$$

$$\leq \sqrt{d(x,y)} + \sqrt{d(y,z)}$$

Hence, $\sqrt{d}$ is a metric on M.

Problem 5 : Let (M, d) be a metric space. Define $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that d_1 is a metric on M.

Solution : $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0\ [since, d(x,y) \geq 0]$

$$d_1(x,y) = 0 \Leftrightarrow \frac{d(x,y)}{1+d(x,y)} = 0$$

$$\Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y. [\because d\ is\ a\ metric]$$

$$d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$$

$$= \frac{d(y,x)}{1+d(y,x)}$$

$$= d_1(y,x)$$

Now, let $x, y, z \in M$.

Then $d_1(x,z) = \frac{d(x,z)}{1+d(x,z)}$

$$= 1 - \frac{1}{1+d(x,z)}$$

$$\leq 1 - \frac{1}{1+d(x,y)+d(y,z)}$$

$$= \frac{1+d(x,y)+d(y,z)-1}{1+d(x,y)+d(y,z)}$$

$$= \frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}$$

$$= \frac{d(x,y)}{1+d(x,y)+d(y,z)} + \frac{d(y,z)}{1+d(x,y)+d(y,z)}$$

$$\leq \frac{d(x,y)}{1+d(x,y)} + \frac{d(y,z)}{1+d(y,z)}$$

$$= d_1(x,y) + d_1(y,z).$$

Thus, $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$.

Hence, d_1 is a metric on M.

Problem 6 : Let (M,d) be a metric space. Define $d_1(x,y) = \min\{1, d(x,y)\}$. Prove that d_1 is a metric on M.

Solution : $d_1(x,y) = \min\{1, d(x,y)\} \geq 0$.

$$\therefore d_1(x,y) \geq 0.$$

$$d_1(x,y) = 0 \Leftrightarrow \min\{1, d(x,y)\}\, 0$$

$$\Leftrightarrow d(x,y) = 0$$

$$\Leftrightarrow x = y.$$

Also, $d_1(x,y) = \min\{1, d(x,y)\}$

$$= \min\{1, d(y,x)\}$$

$$= d(y,x).$$

Now, let $x, y, z \in M$.

Then $d_1(x,z) = \min\{1, d(x,z)\} \leq 1$.

To prove : $d_1(x,z) \leq d_1(x,y) + d_1(y,z)$.

If $d_1(x,y) = 1 \; or \; d_1(y,z) = 1$ the inequality is obvious.

Let $d_1(x,y) < 1$ and $d_1(y,z) < 1$.

Then, $d_1(x,y) + d_1(y,z) = \min\{1, d(x,y)\} + \min\{1, d(y,z)\}$

$$= d(x,y) + d(y,z)$$

$$\geq d(x,z)$$

$$\geq \min\{1, d(x,z)\}$$

$$= d_1(x,z).$$

Thus, $d_1(x,z) \leq d_1(x,y) + d_1(x,z)$.

$\therefore$ d_1 is a metric on M.

Problem 7 : Let M be a non-empty set. Let $d: M \times M \to R$ be a function such that

$(i) d(x,y) = 0$ iff $x = y$.

$(ii) d(x,y) \leq d(x,z) + d(y,z)$ $for\ all\ x,y,z \in M$.

Prove that d is a metric on M.

Solution : Put $y = x$ in (ii).

We have, $d(x,x) \leq d(x,z) + d(x,z)$.

$\therefore 0 \leq 2d(x,z)$ by (i)

$$\therefore d(x,z) \geq 0.$$

Now to prove $d(x,y) = d(y,x)$.

Put $z = x$ in (ii) w get $d(x,y) \leq d(x,x) + d(y,x)$.

$$i.e., d(x,y) \leq d(y,x)\ \ using\ (i)$$

Since this is true for all $x, y \in M$ we have $d(y,x) \leq d(x,y)$.

Hence, $d(x,y) = d(y,x)$.

Now (ii) can be written as $d(x,y) \leq d(x,z) + d(z,y)$ which is the triangle inequality.

$\therefore$ d is a metric on M.

Problem 8 : If $(M_1, d_1), (M_2, d_2), , \ldots, (M_n, d_n)$ are metric spaces then $M_1 \times M_2 \times \ldots \ldots \ldots \ldots \times M_n$ is a metric space with metric d defined by $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)$ where $x = (x_1, x_2, \ldots, x_n); y = (y_1, y_2, \ldots \ldots, y_n)$.

Solution : $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i) \geq 0$.

Also, $d(x,y) = 0 \Leftrightarrow \sum_{i=1}^{n} d_i(x_i, y_i) = 0$

$$\Leftrightarrow d_i(x_i, y_i) = 0 for\ all\ i = 1,2, \ldots \ldots . . n.$$

$$\Leftrightarrow x_i = y_i for\ all\ i = 1,2, \ldots \ldots . . n.$$

$$\Leftrightarrow (x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots \ldots, y_n)$$

$$\Leftrightarrow x = y.$$

Also, $d(x,y) = \sum_{i=1}^{n} d_i(x_i, y_i)$

$$= \sum_{i=1}^{n} d_i(y_i, x_i)$$

$$= d(y,x).$$

Now, let $x, y, z \in M$.

$$d(x,z) = \sum_{i=1}^{n} d_i(x_i, z_i)$$

$$\leq \sum_{i=1}^{n} [d_i(x_i, y_i) + d_i(y_i, z_i)]$$

$$= \sum_{i=1}^{n} d_i(x_i, y_i) + \sum_{i=1}^{n} d_i(y_i, z_i)$$

$$= d(x,y) + d(y,z).$$

$$\therefore d(x,z) \leq d(x,y) + d(y,z).$$

Hence, d is a metric on M.

Problem 9: In a metric space (M,d) prove that $|d(x,z) - d(y,z)| \leq d(x,y)$ for all $x, y, z \in M$.

Solution : let $x, y, z \in M$.

We have $d(x,z) \leq d(x,y) + d(y,z)$.

$\therefore d(x,z) - d(y,z) \leq d(x,y)$. .... (i)

Interchanging x and y in (i) we get

$$d(y,z) - d(x,z) \leq d(y,x) = d(x,y)$$

$\therefore d(y,z) - d(x,z) \leq d(x,y)$. .... (ii)

From (i) and (ii) we get $|d(x,z) - d(y,z)| \leq d(x,y)$.

BOUNDED SETS IN A METRIC SPACE

Definition : Let (M,d) be a metric space. We say that a subset A of M is **bounded** if there exista a positive real numbr k such that $d(x,y) \leq k$ for all $x, y \in A$.

Example 1 : Any finite subset A of a metric space (M,d) is bounded.

Proof : Let A be any finite subset of M.

If $A = \emptyset$, then A is obviously bounded.

Let $A \neq \emptyset$. Then $\{d(x,y)/\ x,y \in A\}$ is a finite set of real numbers.

Let $k = \max\{d(x,y)/x,y \in A\}$.

Clearly, $d(x,y) \leq k$ for all $x,y \in A$.

$\therefore$ A is bounded.

Example 2 : [0, 1] is a bounded subset of R with usual metric since $d(x,y) \leq 1$ for all $x,y \in [0,1]$.

More generally any finite interval and any subset of R which is contained in a finite interval are bounded subsets of R.

Example 3 : $(0,\infty)$ is an unbounded subset of R.

Example 4 : If we consider R with discrete metric, then $(0,\infty)$ is bounded subsets of R, since $d(x,y) \leq 1$ for all $x,y \in (0,\infty)$.

More generally, any subset of a discrete metric space M is a bounded subset of M.

Example 5 : In l_2 let $e_1 = (1,0,0,\dots,0,..), e_2 = (0,1,0,\dots\dots,0,\dots..), e_3 = (0,0,1,0,\dots,0.\dots.)$, ... Let $A = \{e_1,e_2,e_2,\dots\dots,e_n,\dots\dots\dots\}$. Then A is a bounded subset of l_2.

Proof : $d(e_n,e_m) = \begin{cases}\sqrt{2}\ if\ n \neq m \\ 0\ if\ n = m.\end{cases}$

$d(e_n,e_m) = \sqrt{2}$ for all $e_n,e_m \in A$.

$\therefore A$ is a bounded set in l_2.

Example 6 : Let (M,d) be a metric space. Define $d_1(x,y) = \frac{d(x,y)}{1+d(x,y)}$.

We know that (M,d_1) is also a metric space.

Also, $d_1(x,y) < 1$ for all $x,y \in M$.

Hence, (M,d_1) is a bounded metric space.

Definition : Let (M, d) be a metric space. Let $A \subseteq M$. Then the diameter of A, denoted by d(A), is defined by $d(A) = l.u.b.\{d(x,y)/x,y \in A\}$.

Note 1 : A non-empty set A is a bounded set iff d(A) is finite.

Note 2 : Let $A,B \subseteq M$. Then $A \subseteq B \Rightarrow d(A) \leq d(B)$.

Example 1 : The diameter of any non-empty subset in a discrete metric space is 1.

Example 2 : In R the diameter of any interval is equal to the length of the interval. For example, the diameter of [0, 1] is 1.

Example 3 : In any metric space $d(\emptyset) = -\infty$.

OPEN BALL (OPEN SPHERE) IN A METRIC SPACE

Definition : Let (M,d) be a metric space. Let $a \in M$ and r be a positive real number. Then the **open ball** or the open sphere with centre a and radius r denoted by $B_d(a,r)$ is the subset of M given by $B_d(a,r) = \{x \in M \,/\, d(a,x) < r\}$.

When the metric d under consideration is clear we write $B(a,r)$ instead of $B_d(a,r)$.

Note 1 : $B(a,r)$ is always non-empty since it contains atleast its centre a.

Note 2 : $B(a,r)$ is a bounded set.

For, let $x, y \in B(a,r)$.

$x \in B(a,r) \Rightarrow d(a,x) < r$

$y \in B(a,r) \Rightarrow d(a,y) < r$

$$\therefore d(x,y) \le d(x,a) + d(a,y) < r + r = 2r.$$

Thus, $d(x,y) < 2r$.

Hence, $B(a,r)$ is bounded.

Example 1 : Consider R with usual metric. Let $a \in R$.

$$\begin{aligned} \text{Then } B(a,r) &= \{x \in R/\, d(a,x) < r\} \\ &= \{x \in R \,/|x-a| < r\} \\ &= \{x \in R \,/-r < x-a < r\} \\ &= \{x \in R \,/a-r < x < a+r\} \\ &= (a-r, a+r). \end{aligned}$$

Example 2 : Consider C with usual metric. Let $a \in C$.

$$\begin{aligned} \text{Then } B(a,r) &= \{z \in C/\, d(a,z) < r\} \\ &= \{z \in C \,/|z-a| < r\} \end{aligned}$$

This is the interior of the circle with centre a and radius r.

Example 3 : In R^2 with usual metric B(a, r) is the interior of the circle with centre a and radius r.

Example 4 : Let d be the discrete metric on M. Then $B(a,r) = \begin{cases} M \; if \; r > 1 \\ \{a\} \; if \; r \le 1 \end{cases}$.

Proof : We have, $d(x,y) = \begin{cases} 1 \; if \; x \ne y \\ 0 \; if \; x = y \end{cases}$

Let $a \in M$. Let r be any positive real number.

Case (i) $Let\ r > 1$. Then, $B(a,r) = \{x \in M/d(a,x) < r\}$.

Clearly every point $x \in M$ such that $d(a,x) < r$.

Hence, $B(a,r) = M$.

Case (ii) Let $r \leq 1$.

In this case for any point $x \neq a, d(a,x) = 1 \geq r$.

Hence, $x \notin B(a,r)$ so that $B(a,r) = \{a\}$.

$$\therefore B(a,r) = \begin{cases} M\ if\ r > 1 \\ \{a\}\ if\ r \leq 1 \end{cases}.$$

Example 5 : Consider $M = [0,1]$ with usual metric $d(x,y) = |x - y|$.

Here, $B\left(0,\frac{1}{2}\right) = \left\{x \in [0,1]/\ d(0,x) < \frac{1}{2}\right\}$

$$= \left\{x \in [0,1]/|x| < \frac{1}{2}\right\}$$

$$= \left[0,\frac{1}{2}\right).$$

Example 6 : Consider R^2 with the metric d given by

$$d\big((x_1,y_1),(x_2,y_2)\big) = |x_1 - x_2| + |y_1 - y_2|$$

$$Then\ B\big((0,0),1\big) = \{(x,y) \in R^2/\ |x - 0| + |y - 0| < 1\}$$

$$= \{(x,y) \in R^2/\ |x| + |y| < 1\}$$

This is the interior of the square bounded by the four lines $x + y = 1;\ -x + y = 1; -x - y = 1; x - y = 1.\ \ i.e., x + y = 1;\ -x + y = 1; x + y = -1; x - y = 1.$

Example 7 : Consider R^2 with the metric d given by

$$d\big((x_1,y_1),(x_2,y_2)\big) = \max\{|x_1 - x_2| + |y_1 - y_2|\}$$

$$Then\ B\big((0,0),1\big) = \{(x,y) \in R^2/\max\{|x - 0| + |y - 0|] < 1\}$$

$$= \{(x,y) \in R^2/\max\{\ |x| + |y|\} < 1\}$$

This is the interior of the square with vertices $(1,1), (-1,1), (-1,-1) and\ (1,-1)$.

Exercises

1. In R with usual metric find

   (i) $B(-1,1)$ (ii) B(1,1) (iii) B(1/2 , 1)

2. In [0,1] with usual metric find

   (i) B(1/2,1) (ii) B(0,1/4) (iii) B(1,1/2) (iv(B(1/4, ¼)

OPEN SETS

Definition : Let (M,d) be a metric space. Let A be a subset of M. Then A is said to be **open** in M if for every $x \in A$ there exists a positive real number r such that $B(x,r) \subseteq A$.

Example 1 : In R with ususal metric (0, 1) is an open set.

Proof : Let $x \in (0,1)$.

Choose $r = \min\{x - 0, 1 - x\} = \min\{x, 1 - x\}$.

Clearly, $r > 0$ and $B(x,r) = (x - r, x + r) \subseteq (0,1)$.

$\therefore$ (0,1) is open.

Example 2 : In R with usual metric [0, 1) is not open since no open ball with centre 0 is containd in [0,1).

Example 3 : Consider M=[0,2) with usual metric. Let $A = [0,1) \subseteq M$. Then A is open in M.

Proof : Let $x \in [0,1)$

If $x = 0$ then $B\left(0,\frac{1}{2}\right) = \left[0,\frac{1}{2}\right) \subseteq A$.

If $x \neq 0$ choose $r = \min\{x, 1 - x\}$.

Clearly $r > 0$ and $B(x.r) = (x - r, x + r) \subseteq [0,1)$.

$\therefore A$ is open in M.

Example 4 : Any open interval (a,b) is an open set in R with usual metric.

Proof : Let $x \in (a,b)$.

Choose $r = \min\{x - a, b - x\}$

Clearly, $r > 0$ and $B(x,r) \subseteq (a,b)$.

$\therefore (a,b)$ is an open set.

Note : Similarly we can prove that $(-\infty, a)$ and (a, ∞) are open sets.

Example 5 : In R with usual metric, the set {0} is not an open set since, any open ball with centre 0 is not contained in {0}.

Example 6 : In R with usual metric any finite non-empty subset A of R is not an open set.

Proof : Any open ball in R is a bounded open interval which is an infinite subset of R.

Hence, it cannot be contained in the finite subset A.

Hence, A is not open in R.

Example 7 : Q is not open in R.

Proof : Let $x \in Q$.

Then, for any $r > 0$ the interval $(x - r, x + r)$ contains both rational and irrational numbers.

$\therefore (x - r, x + r)$ is not a subset of Q.

Hence, Q is not open in R.

Example 8 : The set of irrational numbers I is not open in R.

Proof : Let $x \in I$.

Then, for any $r > 0$ the interval $(x - r, x + r)$ contains both rational and irrational numbers.

$\therefore (x - r, x + r)$ is not a subset of I.

Hence, I is not open in R.

Example 9 : Z is not open in R.

Proof : Let $x \in Z$.

Then, for any $r > 0$ the interval $(x - r, x + r)$ is not a subset of Z.

Hence, Z is not open in R.

Example 10 : In a discrete metric space M every subset A is open.

Proof : If $A = \emptyset$, trivially A is open.

Let $A \neq \emptyset$.

Let $x \in A$.

Then $B\left(x, \frac{1}{2}\right) = \{x\} \subseteq A$.

Hence, A is open in M.

Theorem 1.1 : In any metric space M, (i) $\emptyset$ is open.

(ii) M is open.

Proof : (i) Trivially $\emptyset$ is an open set.

(ii) Let $x \in M$. Clearly for any $r > 0, B(x, r) \subseteq M$. Hence, M is an open set.

Theorem 1.2 : In any metric space (M,d) each open ball is an open set.

Proof : Let $B(a, r)$ be an open ball in M.

Let $x \in B(a, r)$

Then $d(a, x) < r$.

$$\therefore 0 < r - d(a, x).\ i.e., r - d(a, x) > 0.$$

Let $r_1 = r - d(a,x)$.

We claim that $B(x,r_1) \subseteq B(a,r)$.

Let $y \in B(x,r_1)$

$$\therefore d(x,y) < r_1 = r - d(a,x).$$

$\therefore d(x,y) + d(a,x) < r$. .... (1)

Now, $d(a,y) \leq d(a,x) + d(x,y) < r$ $[From\ (1)]$

$$\therefore d(a,y) < r.$$

$$\therefore y \in B(a,r).$$

Hence, $B(x,r_1) \subseteq B(a,r)$.

$\therefore B(a,r)$ is an open set.

Theorem 1.3 : In any metric space the union of family of open sets is open.

Proof : Let (M,d) be a metric space,

Let $\{A_i / i \in I\}$ be a family of open sets in M.

Let $A = \bigcup_{i \in I} A_i$.

If $A = \emptyset$ then A is open.

$\therefore$ Let $A \neq \emptyset$.

Let $x \in A$. Then $x \in A_i\ for\ some\ i \in I$.

Since A_i is open, there exists an open ball B(x, r) such that $B(x,r) \subseteq A_i$.

$$\therefore B(x,r) \subseteq A.$$

Hence A is open.

Theorem 1.4 : In any metric space the intersection of a finite number of open sets is open.

Proof : Let (M,d) be a metric space.

Let $A_1, A_2, \ldots \ldots \ldots, A_n$ be open sets in M.

Let $A = A_1 \cup A_2 \cup \ldots \ldots \ldots \cup A_n$.

If $A = \emptyset$ then A is open.

$\therefore$ Let $A \neq \emptyset$.

Let $x \in A$. Then $x \in A_i\ for\ each\ i = 1,2, \ldots \ldots . . n$.

Since A_i is open, there is a positive real number r_i such that $B(x,r_i) \subseteq A_i$. ....(1)

Let $r = \min\{r_1, r_2, \ldots \ldots \ldots \ldots . . , r_n\}$

Obviously r is a positive real number and $B(x,r) \subseteq B(x,r_i)$ for all $i = 1,2, \ldots \ldots . , n$.

Hence $B(x,r) \subseteq A_i\ for\ all\ i = 1,2, \ldots \ldots . , n\ [from\ (1)]$.

$$\therefore B(x,r) \subseteq \bigcap_{i=1}^{n} A_i.$$

$$\therefore B(x,r) \subseteq A.$$

$\therefore A$ is open.

Note : The intersection of an infinite number of open sts in a metric space need not be open.

For example, Consider R with usual metric.

Let $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$.

Then A_n is open in R for all n.

But $\bigcap_{n=1}^{\infty} A_n = \{0\}$ which is not open in R.

Characterization of open sets in terms of open balls

Theorem 1.5 : Let (M,d) be a metric space. Let A be any non-empty subset of M. Then A is open iff A can be expressed as th union of family of open balls.

Proof : Let A be any non-empty subset of M.

Assume that A is open.

To prove, A can be expressed as the union of family of open balls.

Let $x \in A$.

Since, A is an open set there exists an open ball $B(x, r_x)$ such that $B(x, r_x) \subseteq A$.

Clearly, $\bigcup_{x \in A} B(x, r_x) = A$.

Thus A is the union of family of open balls.

Conversely, Assume that A can be expressed as the union of family of open balls.

To prove, A is open.

By Theorem 1.2, each open ball is an open set.

By Theorem 1.3, In any metric space, the union of family of open sets is open.

Hence, A is open.

SOLVED PROBLEMS

Problem 1 : Let (M,d) be a metric space. Let x,y be two distinct points in M. Prove that there exists two disjoint open balls with centres x and y respectively.

Solution : Since, $x \neq y, d(x,y) = r > 0$.

Consider the open balls $B\left(x, \frac{1}{4}r\right)$ and $B\left(y, \frac{1}{4}r\right)$.

We claim that $B\left(x, \frac{1}{4}r\right) \cap B\left(y, \frac{1}{4}r\right) = \emptyset$.

Suppose $B\left(x, \frac{1}{4}r\right) \cap B\left(y, \frac{1}{4}r\right) \neq \emptyset$.

Let $z \in B\left(x, \frac{1}{4}r\right) \cap B\left(y, \frac{1}{4}r\right)$

$$\therefore z \in B\left(x, \frac{1}{4}r\right) \text{ and } z \in B\left(y, \frac{1}{4}r\right)$$

$$\therefore d(x,z) < \frac{1}{4}r \text{ and } d(y,z) < \frac{1}{4}r.$$

Now, $d(x,y) \leq d(x,z) + d(z,y) < \frac{1}{4}r + \frac{1}{4}r = \frac{1}{2}r$.

which is a contradiction.

Hence, $B\left(x, \frac{1}{4}r\right) \cap B\left(y, \frac{1}{4}r\right) = \emptyset$.

Problem 2 : Let (M, d) be a metric space. Let $x \in M$. Show that $\{x\}^c$ is open.

Solution : Let $y \in \{x\}^c$. Then $y \neq x$.

$\therefore d(x,y) = r > 0$.

Clearly $B\left(y, \frac{1}{2}r\right) \subseteq \{x\}^c$.

$\therefore$ $\{x\}^c$ is open.

Problem 3 : Let (M,d) be a metric space. Show that every subset of M is open iff $\{x\}$ is open for all $x \in M$.

Solution : Suppose every subset of M is open.

Then obviously $\{x\}$ is open for all $x \in M$.

Conversely, assume that $\{x\}$ is open for all $x \in M$.

To prove A is open.

Let A be any subset of M.

If $A = \emptyset$ then A is open.

$\therefore$ Let $A \neq \emptyset$. Then $A = \bigcup_{x \in A}\{x\}$.

By hypothesis, $\{x\}$ is open.

Since arbitrary union of open sets is open, A is open.

Problem 4 : Let $A = \left\{(a_n)/(a_n) \in l_2 \text{ and } [\sum_{n=1}^{\infty} {a_n}^2]^{\frac{1}{2}} < 1\right\}$. Prove that A is open in l_2.

Solution : We first prove that $A = B(\mathbf{0}, 1)$ where $\mathbf{0} = (0,0, \dots\dots..)$.

Let $x \in A$. Hence, $\sum_{n=1}^{\infty} {x_n}^2 < 1$.

$$\therefore d(x, \mathbf{0}) = \left[\sum_{n=1}^{\infty} (x_n - 0)^2\right]^{1/2} = \left[\sum_{n=1}^{\infty} (x_n)^2\right]^{1/2} < 1.$$

Thus, $d(x, \mathbf{0}) < 1$.

$$\therefore x \in B(\mathbf{0}, 1)$$

$$\therefore A \subseteq B(\mathbf{0}, 1) \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

Now, let $y \in B(\mathbf{0}, 1)$

$$\therefore d(\mathbf{0}, y) < 1.$$

$$\therefore \left[\sum_{n=1}^{\infty}(y_n - 0)^2\right]^{\frac{1}{2}} < 1$$

$$\therefore \left[\sum_{n=1}^{\infty}(y_n)^2\right]^{1/2} < 1.$$

$$\therefore y \in A.$$

$$\therefore B(\mathbf{0}, 1) \subseteq A \ \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

From (1) and (2) we get $A = B(\mathbf{0}, 1)$.

Now the open ball $B(\mathbf{0}, 1)$is an open set.

Hence, A is an open set.

Problem 5 : Prove that any open subset of R can be expressed as the union of a countable number of mutually disjoint open intervals.

Solution : Let A be an open subset of R.

Let $x \in A$.

Then there exists a positive real number r such that $B(x, r) = (x - r, x + r) \subseteq A$.

Thus there exists an open interval I such that $x \in I$ and $I \subseteq A$.

Let I_x denote the largest open interval such that $x \in I$ and $I_x \subseteq A$.

Clearly, $\bigcup_{x \in A} I_x = A$.

We claim that $I_x = I_y \ or \ I_x \cap I_y = \emptyset$.

Suppose $I_x \cap I_y \neq \emptyset$.

Then $I_x \cup I_y$ is an open interval contained in A.

But I_x is the largest open interval such that $x \in I_x \ and \ I_x \subseteq A$.

$\therefore I_x \cup I_y = I_x$ so that $I_y \subseteq I_x$.

Similarly, $I_x \subseteq I_y$.

$\therefore I_x = I_y$. Thus the intervals I_x are mutually disjoint.

We claim that the set $F = \{I_x / \ x \in A\}$ is countable.

Now for each $I_x \in F$ choose a rational number $r_x \in I_x$.

Since the intervals I_x are mutually disjoint $I_x \neq I_y \Rightarrow r_x \neq r_y$.

$\therefore f: F \rightarrow Q$ defined by $f(I_x) = r_x$ is 1-1.

$\therefore F$ is equivalent to a subset of Q which is countable.

$\therefore F$ is countable.

EQUIVALENT METRICS

Definition : Let d and ρ be the two metrics on M. Then the metrics d and ρ are said to be equivalent if the open sets of (M, ρ) are the open sets of (M, d) and conversely.

Problem 6 : Let (M, d) be a metric space. Define $\rho(x, y) = 2d(x, y)$. Then d and ρ are equivalent metrics.

Solution : We know that ρ is a metric on M.

We first prove that $B_d(a, r) = B_\rho(a, 2r)$.

Let $x \in B_d(a, r)$

$$\therefore d(a, x) < r.$$

$$\therefore 2d(a, x) < 2r.$$

$$\therefore \rho(x, y) < 2r.$$

Hence. $x \in B_\rho(a, 2r)$.

$$\therefore B_d(a, r) \subseteq B_\rho(a, 2r) \quad \ldots\ldots\ldots\ldots (1)$$

Now, let $x \in B_\rho(a, 2r)$

$$\therefore \rho(a, x) < 2r.$$

$$\therefore \frac{1}{2}\rho(a, x) < r.$$

$$\therefore d(a, x) < r.$$

$$\therefore x \in B_d(a, r)$$

$$\therefore B_\rho(a, 2r) \subseteq B_d(a, r) \quad \ldots\ldots\ldots\ldots (2)$$

$$\therefore By\ (1) and\ (2) we\ get, B_d(a, r) = B_\rho(a, 2r) \ldots\ldots\ldots\ldots (3)$$

Now, let G be any open subset in (M,d). Let $a \in G$.

Hence, there exists $r > 0$ such that $B_d(a, r) \subseteq G$.

$$\therefore B_\rho(a, 2r) \subseteq G \ \big(using\ (3)\big)$$

$\therefore G$ is open in (M, ρ).

Conversely, suppose G is open in (M, ρ).

Let $a \in G$.

Hence, there exists $r > 0$ such that $B_\rho(a,r) \subseteq G$.

$$\therefore B_d\left(a, \frac{1}{2}r\right) \subseteq G \ \left(using\ (3)\right)$$

$\therefore$ G is open in (M,d).

$\therefore$ $d\ and\ \rho$ are equivalent metrics.

Problem 7 : Let (M,d) be a metric space. Define $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$. Prove that $d\ and\ \rho$ are equivalent metrics on M.

Solution : We know that ρ is a metric on M.

We first prove that $B_\rho(a,r) = B_d\left(a, \frac{r}{1-r}\right)\ provided\ 0 < r < 1$.

Let $x \in B_\rho(a,r)$

$$\therefore \rho(a,x) < r.$$

$$\therefore \frac{d(x,y)}{1+d(x,y)} < r.$$

$$\therefore d(x,y) < r\big(1 + d(x,y)\big) = r + r\, d(x,y).$$

$$\therefore d(x,y) - rd(x,y) < r$$

$$\therefore d(x,y)(1-r) < r$$

$$\therefore d(x,y) < \frac{r}{1-r}\ (since\ 0 < r < 1)$$

Hence. $x \in B_d\left(a, \frac{r}{1-r}\right)$.

$$\therefore B_\rho(a,r) \subseteq B_d\left(a, \frac{r}{1-r}\right) \quad \ldots\ldots\ldots\ldots(1)$$

Now, let $x \in B_d\left(a, \frac{r}{1-r}\right)$

$$\therefore d(a,x) < \frac{r}{1-r}$$

$$\therefore (1-r)d(a,x) < r$$

$$\therefore d(a,x) - rd(a,x) < r$$

$$\therefore d(a,x) < r + rd(a,x)$$

$$\therefore d(a,x) < r\big(1 + d(a,x)\big)$$

$$\therefore \frac{d(a,x)}{\big(1 + d(a,x)\big)} < r$$

$$\therefore \rho(a,x) < r$$

$$\therefore x \in B_\rho(a,r)$$

$$\therefore B_d\left(a, \frac{r}{1-r}\right) \subseteq B_\rho(a,r) \quad \dots\dots\dots (2)$$

$$\therefore By\ (1) and\ (2) we\ get, B_\rho(a,r) = B_d\left(a, \frac{r}{1-r}\right) \quad \dots\dots\dots (3)$$

Now, let G be any open subset in (M, ρ). Let $a \in G$.

Hence, there exists $r > 0$ such that $B_\rho(a,r) \subseteq G$.

Without loss of generality we may assume that $r < 1$.

$$\therefore B_d\left(a, \frac{r}{1-r}\right) \subseteq G\ \left(using\ (3)\right)$$

$\therefore G$ is open in (M, d).

Conversely, suppose G is open in (M, d).

Let $a \in G$.

Hence, there exists $r > 0$ such that $B_d(a,r) \subseteq G$.

$$\therefore B_\rho\left(a, \frac{r}{1-r}\right) \subseteq G\ \left(using\ (3)\right)$$

$\therefore G$ is open in (M, ρ).

$\therefore d\ and\ \rho$ are equivalent metrics.

Problem 8 : If $d\ and\ \rho$ are metrics on M and if there exists $k > 1$ such that $\frac{1}{k}\rho(x,y) \leq d(x,y) \leq k\rho(x,y)$ for all $x, y \in M$. Prove that $d\ and\ \rho$ are equivalent metrics.

Solution : Suppose that there exists $k > 1$ such that $\frac{1}{k}\rho(x,y) \leq d(x,y) \leq k\rho(x,y)$ .... (1) for all $x, y \in M$.

Let G be an open set in (M, d).

Let $a \in G$.

Hence, there exists $r > 0$ such that $B_d(a,r) \subseteq G$. .... (2)

We now claim that $B_\rho\left(a, \frac{r}{k}\right) \subseteq G$

Let $x \in B_\rho\left(a, \frac{r}{k}\right)$

$\therefore \rho(a,x) < \frac{r}{k}$.

$$\therefore k\ \rho(a,x) < r.$$

$$\therefore d(a,x) < r.\ \ [using\ 1]$$

$$\therefore x \in B_d(a,r) \subseteq G\ [By\ (2)]$$

$$\therefore x \in G.$$

Hence, $B_\rho\left(a, \frac{r}{k}\right) \subseteq G$.

$\therefore G$ is open in (M, ρ).

Conversely, let G be open in (M, ρ).

Let $a \in G$.

Hence, there exists $r > 0$ such that $B_\rho(a, r) \subseteq G$ ……………. (3)

We now claim that $B_d\left(a, \frac{r}{k}\right) \subseteq G$

Let $x \in B_d\left(a, \frac{r}{k}\right)$

$\therefore d(a, x) < \frac{r}{k}$.

$$\therefore k\, d(a, x) < r.$$

$$\therefore \rho(a, x) < r. \quad [using\ 1]$$

$$\therefore x \in B_\rho(a, r) \subseteq G\ [By\ (2)]$$

$$\therefore x \in G.$$

Hence, $B_d\left(a, \frac{r}{k}\right) \subseteq G$.

$\therefore G$ is open in (M, d).

Hence $d\ and\ \rho$ are equivalent metrics on M.

Exercises

1. Determine which of the following subsets of R are open in R with usual metric.

   (i) R (ii) N (iii) Z (iv) Q (v) (1,2)U(3,4)

   (vi) $(0, \infty)$ (vii) (-∞, a)

2. Prove that the complement of any finite subset of a metric space M is open.

SUBSPACE

Definition : Let (M, d) be a metric space. Let M_1 be a non-empty subset of M. Then M_1 is also a metric space with the same metric d. We say that (M_1, d) is a subspace of (M, d).

Note : If M_1 is a subspace of M a set which is open in M_1 need not be open in M.

For example, if M=R with usual metric and $M_1 = [0,1]$ then $\left[0, \frac{1}{2}\right)$ is open in M_1 but not open in M.

Theorem 1.6 : Let M be a metric space and M_1 a subspace of M. Let $A_1 \subseteq M_1$. Then A_1 is open in M_1 iff there exists an open set A in M such that $A_1 = A \cap M_1$.

Proof : Let M_1 be a subspace of M.

We denote $B_1(a,r)$ the open ball in M_1 with centre a and radius r.

Then $B_1(a,r) = \{x \in M_1 \,/\, d(a,x) < r\}$.

Also, $B(a,r) = \{x \in M \,/\, d(a,x) < r\}$.

Hence, $B_1(a,r) = B(a,r) \cap M_1$. .... (1)

Now, let A_1 be an open set in M_1.

$A_1 = \bigcup_{x \in A_1} B_1\big(x, r(x)\big)$ $\quad by\ theorem\ 1.5$

$$= \bigcup_{x \in A_1} [B(x, r(x)) \cap M_1] \quad by\ (1)$$

$$= [\bigcup_{x \in A_1} B(x, r(x))] \cap M_1$$

$= A \cap M_1\ where\ A = \bigcup_{x \in A_1} B(x, r(x))$ which is open in M.

Conversely, let $A_1 = A \cap M_1$ where A is open in M.

We claim that A_1 is open in M_1.

Let $x \in A_1$.

$$\therefore x \in A\ and\ x \in M_1.$$

Since A is open in M there exists a positive real number r such that $B(x,r) \subseteq A$.

$$\therefore M_1 \cap B(x,r) \subseteq M_1 \cap A.$$

$$i.e., B_1(x,r) \subseteq A_1\ [using\ (1)]$$

Hence, A_1 is open in M_1.

Example 1 : Let $M = R$ and $M_1 = [0,1]$. Let $A_1 = \left[0, \frac{1}{2}\right)$.

Now, $A_1 = \left[0, \frac{1}{2}\right) = \left(-\frac{1}{2}, \frac{1}{2}\right) \cap [0,1]$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is open in R.

$\therefore \left[0, \frac{1}{2}\right)$ is open in [0,1].

Example 2 : Let $M = R$ and $M_1 = [1,2] \cup [3,4]$.

Let $A_1 = [1,2]$. Then $A_1 = [1,2] = \left(\frac{1}{2}, \frac{5}{2}\right) \cap M_1$.

$\therefore$ [1,2] is open in M_1.

Similarly, [3,4] is open in M_1.

Hence, $[1,2] \cup [3,4]$ is open in M_1.

Problem 1 : Let M_1 be a subspace of a metric space M. Prove that every open set A_1 of M_1 is open in M iff M_1 itself is open in M.

Solution : Suppose every open set A_1 of M_1 is open in M.

Now, M_1 is open in M_1.

Hence, M_1 is open in M.

Conversely, suppose M_1 is open in M.

Let A_1 be an open set in M_1.

Then by theorem 1.6, there exists an open set A in M such that $A_1 = A \cap M_1$.

Since A and M_1 are open in M_1 we get A_1 is open in M.

INTERIOR OF A SET

Definition : Let (M, d) be a metric space. Let $A \subseteq M$. Let $x \in A$. Then x is said to be an interior point of A if there exists a positive real number r such that $B(x, r) \subseteq A$.

The set of all interior points of A is called the interior of A and it is denoted by IntA.

Note : $IntA \subseteq A$.

Example 1 : Consider R with usual metric.

(a) Let $A = [0,1]$. Clearly 0 and 1 aree not interior points of A and any point $x \in (0,1)$ is an interior point of A. Hence, $IntA = (0,1)$.

(b) Let $A = Q$. Let $x \in Q$.

Then for any positive real number r, $B(x, r) = (x - r, x + r)$ contains irrational numbers.

$\therefore$ $B(x, r)$ is not a subset of Q.

$\therefore$ x is not an interior point of Q.

Since $x \in Q$ is arbitrary, no point of Q is an interior point.

$$\therefore IntQ = \emptyset.$$

(c) Let A be a finite subset of R. Then $IntA = \emptyset$.

(d) Let $A = \left\{0, 1, \frac{1}{2}, \dots\dots\dots., \frac{1}{n}, \dots..\right\}$. Then $IntA = \emptyset$.

Example 2 : Consider R with discrete metric.

Let $A = [0,1]$. $Let\ x \in [0,1]$.

Then $B\left(x, \frac{1}{2}\right) = \{x\} \subseteq A$.

$\therefore$ x is an interior point of A.

Since, $x \in [0,1]$ is arbitrary, IntA=A.

Example 3 : In a discrete metric space M, IntA=A for any subset A of M.

Basic properties of interior

Theorem 1.7 : Let (M,d) be a metric space. Let $A, B \subseteq M$.

(i) A is open iff $A = IntA$. In particular, $Int\emptyset = \emptyset$ and $IntM = M$.

(ii) IntA=Union of all open sets contained in A.

(iii) IntA is an open subset of A and if B is any other open set contained in A then $B \subseteq IntA$. i.e., IntA is the largest open set contained in A.

(iv) $A \subseteq B \Rightarrow IntA \subseteq IntB$.

(v) $Int(A \cap B) = IntA \cap IntB$.

(vi) $Int(A \cup B) \supseteq IntA \cup IntB$.

Proof : (i) From the definition of open set, A is open iff $A = IntA$.

Also, $Int\emptyset = \emptyset$ and $IntM = M$.

(ii) Let $G = \cup \{B \,/\, B \text{ is an open subset of } A\}$.

To prove IntA=G.

Let $x \in IntA$.

$\therefore$ There exists a positive real number r such that $B(x,r) \subseteq A$.

Thus $B(x,r)$ is an open set contained in A.

$$\therefore B(x,r) \subseteq G.$$

$$\therefore x \in G.$$

$$\therefore IntA \subseteq G. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Now, let $x \in G$.

Then there exists an open set B such that $x \in B$ and $B \subseteq A$.

Now, since B is open and $x \in B$ there exists a positive real number r such that $B(x,r) \subseteq B \subseteq A$.

$\therefore x$ is an interior point of A.

Hence, $G \subseteq IntA \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$

From (1) and (2) , we get G=IntA.

(iii) Since union of any collection of open sets is open, (ii)⇒IntA is an open set.

Trivially $IntA \subseteq A$.

Now, let B be any open set contained in A.

Then $B \subseteq G = IntA \quad [From\ (ii)]$

$\therefore$ IntA is the largest open set contained in A.

(iv) Let $x \in IntA$.

$\therefore$ There exists a positive real number r such that $B(x,r) \subseteq A$.

But $A \subseteq B$.

Hence, $B(x,r) \subseteq B$.

$\therefore x \in Int\ B$

Hence, $IntA \subseteq IntB$.

(v) We have, $A \cap B \subseteq A$.

$$\therefore Int(A \cap B) \subseteq IntA\ [From\ (iv)]$$

Also, $A \cap B \subseteq B$.

$$\therefore Int(A \cap B) \subseteq IntB\ [From\ (iv)]$$

$$\therefore Int(A \cap B) \subseteq IntA \cap IntB \ldots\ldots\ldots..(1)$$

Now, $IntA \subseteq A; IntB \subseteq B$.

Hence, $IntA \cap IntB \subseteq A \cap B$.

Thus, $IntA \cap IntB\ is\ an\ open\ set\ contained\ in\ A \cap B$.

But Int($A \cap B$) is the largest open set contained in $A \cap B$.

$\therefore IntA \cap IntB \subseteq Int(A \cap B) \ldots\ldots\ldots..(2)$

From (1) and (2) we get $Int(A \cap B) = IntA \cap IntB$.

(vi) We have, $A \subseteq A \cup B$.

$$\therefore IntA \subseteq Int(A \cup B)\ [From\ (iv)]$$

Also, $B \subseteq A \cup B. \therefore IntB \subseteq Int(A \cup B)\ [From\ (iv)]$

$$\therefore IntA \cup IntB \subseteq Int(A \cup B).$$

$$i.e., Int(A \cup B) \supseteq IntA \cup IntB.$$

Note : $Int(A \cup B)\ need\ not\ be\ equal\ to\ IntA \cup IntB$.

For example, in R with usual metric, consider $A = (0,2]\&B = (2,3)$.

Then $A \cup B = (0,3)$.

Clearly, $Int(A \cup B) = (0,3)$.

But $IntA = (0,2)\&\ IntB = (2,3)$

$$IntA \cup IntB = (0,2) \cup (2,3) = (0,3) - \{2\}.$$

$$\therefore Int(A \cup B) \neq IntA \cup IntB.$$

UNIT II

CLOSED SETS

Definition : Let (M,d) be a metric space. Let $A \subseteq M$. Then A is said to be closed in M if the complement of A is open in M.

Example 1 : In R with usual metric any closed interval [a,b] is a closed set.

Proof : $[a,b]^c = R - [a,b] = (-\infty, a) \cup (b, \infty)$.

Also $(-\infty, a) and\ (b, \infty)$ are open in R.

i.e., $[a,b]^c$ is open in R.

Hence, [a,b] is closed in R.

Example 2 : In R with usual metric [a,b) is neither open nor closed.

Proof : [a,b) is not open in R since a is not an interior point of [a,b).

Now, $[a,b)^c = R - [a,b) = (-\infty, a) \cup [b, \infty)$ and this set is not open since b is not an interior point.

$\therefore$ $[a,b)$ is not closed in R.

Hence, [a,b) is neither open nor closed in R.

Example 3 : In R with usual metric (a,b] is neither open not closed.

Proof is similar to example 2.

Example 4 : Z is closed.

Proof : $Z^c = \cup_{n=1}^{\infty}(n, n+1)$.

The open interval (n,n+1) is open and the union of open sets is open.

$\therefore$ Z^c is open.

Hence, Z is closed.

Example 5 : Q is not closed in R.

Proof : Q^c=the set of irrational which is not open in R.

$\therefore$Q is not closed in R.

Example 6 : The set of irrational numbers is not closed in R.

Proof is similar to that of Example 5.

Example 7 : In R with usual metric every singleton set is closed.

Proof : Let $a \in R$.

Then $\{a\}^c = R - \{a\} = (-\infty, a) \cup (a, \infty)$.

Since $(-\infty, a)$ and (a, ∞) are both open sets, $(-\infty, a) \cup (a, \infty)$ is open.

Thus, $\{a\}^c$ is open.

Hence, {a} is closed.

Example 8 : Every subset of a discrete metric space is closed.

Proof : Let (M,d) be a discrete metric space.

Let $A \subseteq M$.

Since, every subset of a discrete metric spac is open. A^c is open.

$\therefore$ A is closed.

Definition : Let (M, d) be a metric space. Let $a \in M$. Let r be any positive real number. Then the closed ball or the closed sphere with centre a and radius r denoted by $B_d[a, r]$, is defined by $B_d[a, r] = \{x \in M / d(a, x) \leq r\}$.

When the metric d under consideration is clear we write $B[a, r]$ instead of $B_d[a, r]$.

Example 1 : In R with usual metric $B[a, r] = [a - r, a + r]$.

Example 2 : In R^2 with usual metric let $a = (a_1, a_2) \in R^2$.

Then $B[a, r] = \{(x, y) \in R^2 / d\big((a_1, a_2)\big), (x, y)) \leq r\}$

$$= \{(x, y) \in R^2 / (x - a_1)^2 + (y - a_2)^2 \leq r\}$$

Hence, $B[a, r]$ is the set of all points which lie within and on the circumference of the circle with centre a and radius r.

Theorem 2.1 : In any metric space every closed ball is a closed set.

Proof : Let (M,d) be a metric space.

Let $B[a, r]$ be an open ball in M.

Case (i) Suppose $B[a, r]^c = \emptyset$

$\therefore$ $B[a, r]^c$ is open and hence B[a,r] is closed.

Case (ii) Suppose $B[a, r]^c \neq \emptyset$

Let $x \in B[a, r]^c$

$\therefore$ $x \notin B[a, r]$

Then $d(a, x) > r$.

$$\therefore d(a, x) - r > 0.$$

Let $r_1 = d(a, x) - r$.

We claim that $B(x, r_1) \subseteq B[a, r]^c$.

Let $y \in B(x, r_1)$

$$\therefore d(x,y) < r_1 = d(a,x) - r.$$

$\therefore d(x,y) + r < d(a,x).$

$\therefore d(a,x) > d(x,y) + r$ .... (1)

Now, $d(a,x) \leq d(a,y) + d(y,x)$

$$\therefore d(a,y) \geq d(a,x) - d(y,x)$$

$$> d(x,y) + r - d(y,x)[From\ (1)] = r$$

$$\therefore d(a,y) > r.$$

$$\therefore y \notin B(a,r).$$

Hence, $y \in B[a,r]^c$

Hence, $B(x, r_1) \subseteq B[a,r]^c$.

$\therefore B[a,r]^c$ is open in M.

$\therefore B[a,r]$ is closed in M.

Theorem 2.2 : In any metric space M, (i) $\emptyset$ is closed, (ii) M is closed.

Proof : Since $M^c = \emptyset$ is open, M is closed.

Similarly, $\emptyset^c = M$ is open and hence $\emptyset$ is closed.

Theorem 2.3 : In any metric space arbitrary intersection of closed sets is closed.

Proof : Let (M, d) be a metric space,

Let $\{A_i / i \in I\}$ be a family of closed sets in M.

Let $A = \bigcap_{i \in I} A_i$.

We claim that A is closed.

We have, $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} {A_i}^c$ $(By\ De - Morgan's\ Law)$

Since A_i is closed, ${A_i}^c$ is open.

$\therefore \bigcup_{i \in I} {A_i}^c$ is open.

$\therefore \left(\bigcap_{i \in I} A_i\right)^c$ is open.

Hence $\bigcap_{i \in I} A_i$ is closed.

Hence A is closed.

Theorem 2.4 : In any metric space the union of a finite number of closed sets is closed.

Proof : Let (M, d) be a metric space.

Let $A_1, A_2, \dots \dots \dots, A_n$ be closed sets in M.

By De-Morgan's law, $(A_1 \cup A_2 \cup \dots \dots . \cup A_n)^c = {A_1}^c \cap {A_2}^c \cap \dots \dots \cap {A_n}^c$.

Since each A_i is closed, ${A_i}^c$ is open.

Since finite intersection of open sets are open, ${A_1}^c \cap {A_2}^c \cap \dots \dots \cap {A_n}^c$ is open.

$\therefore (A_1 \cup A_2 \cup \dots \dots . \cup A_n)^c$ is open.

$\therefore A_1 \cup A_2 \cup \dots \dots . \cup A_n$ is closed.

Note : The union of an infinite number of closed sets in a metric space need not be closed.

For example, Consider R with usual metric.

Let $A_n = \left[\frac{1}{n}, 1\right]$ $where\ n = 1,2,3, \dots \dots \dots$

Then $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = \{1\} \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right] \cup \dots \dots \dots \dots$

$= (0,1]$ which is not closd in R.

Hence, $\bigcup_{n=1}^{\infty} A_n$ is not closed.

Theorem 2.5 : Let M b a metric space and M_1 be a subspace of M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 iff there exists a set F which is closed in M such that $F_1 = F \cap M_1$.

Proof : Let F_1 be closed in M_1.

$\therefore M_1 - F_1$ is open in M_1.

$\therefore M_1 - F_1 = A \cap M_1$ where A is open in M (By theorem 1.6)

Now, $F_1 = M_1 - (A \cap M_1) = M_1 - A = A^c \cap M_1$.

Also, since A is open in M, A^c is closed in M.

$\therefore F_1 = F \cap M_1$ where $F = A^c$ is closed in M.

Proof of the converse is similar.

CLOSURE

Definition : Let A be a subset of a metric space (M,d). The closure of A, denoted by $\bar{A}$ is defined to be the intersection of all closed sets which contain A. Thus,

$$\bar{A} = \cup \{B/B \text{ is closed in M and } A \subseteq B\}.$$

Note : Since intersection of any collection of closed sets is closed, $\bar{A}$ is a closed set. Further, $\bar{A} \supseteq A$. Also if B is any closed set containing A then $\bar{A} \subseteq B$. Thus $\bar{A}$ is the smallest closed set containing A.

Theorem 2.6 : A is closed iff $A = \bar{A}$.

Proof : Suppose $A = \bar{A}$.

Since $\bar{A}$ is closed, A is closed.

Conversely, suppose A is closed. Then the smallest closed set containing A is A itself.

Hence $A = \bar{A}$.

Note : In particular, $(i)\emptyset = \bar{\emptyset}$ $(ii)M = \bar{M}$ $(iii)\bar{\bar{A}} = \bar{A}$.

Example 1: Consider R with usual metric.

(a) Let $A = [0,1]$. We know that A is a closed set.

$$\therefore \bar{A} = A = [0,1].$$

(b) Let A=(0,1). Then [0,1] is a closed set containing (0,1). Obviously [0,1] is the smallest closed set containing (0,1).

$$\therefore \bar{A} = [0,1].$$

Example 2 : In a discrete metric space (M,d) any subset A of M is closed. Hence $\bar{A} = A$.

Theorem 2.7 : Let (M,d) be a metric space. Let $A, B \subseteq M$.

Then (i) $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$

(ii) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(iii) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof : **(i)** Let $A \subseteq B$

Now $\bar{B} \supseteq B \supseteq A$.

$\therefore \bar{B}$ is a closed set containint A.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subseteq \bar{B}$.

(ii) We have $A \subseteq A \cup B$.

$\therefore \bar{A} \subseteq \overline{A \cup B}$.

Similarly, $\bar{B} \subseteq \overline{A \cup B}$

$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$. .... (1)

Now $\bar{A}$ is a closed set containing A and $\bar{B}$ is closed set containing B.

$\therefore \bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$.

But $\overline{A \cup B}$ *is the smallest closed set containing* $A \cup B$.

$\therefore \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$. .... (2)

From (1) and (2) we get $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(iii) We have, $A \cap B \subseteq A$

$$\therefore \overline{A \cap B} \subseteq \bar{A}$$

Similarly, $A \cap B \subseteq B$

$$\therefore \overline{A \cap B} \subseteq \bar{B}$$

Hence, $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

Note : $\overline{A \cap B}$ need not be equal to $\bar{A} \cap \bar{B}$.

For example, in R with usual metric, take A=(0,1) and B=(1,2).

Then, $A \cap B = \emptyset$.

$$\therefore \overline{A \cap B} = \overline{\emptyset} = \emptyset.$$

But $\bar{A} \cap \bar{B} = [0,1] \cap [1,2] = \{1\}$.

$\therefore \overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

Note : In a metric space (M,d) if $E_1, E_2, \dots\dots.., E_n$ are subsets of M then, $\overline{E_1 \cup E_2 \cup \dots\dots..\cup E_n} = \overline{E_1} \cup \overline{E_2} \cup \dots\dots\dots\dots \cup \overline{E_n}$.

LIMIT POINT

Definition : Let (M,d) be a metric space. Let $A \subseteq M$. Let $x \in M$. Then x is called a limit point or a cluster point or an accumulation point of A if every open ball with centre x contains atleast one point of A different from x.

$$i.e., B(x,r) \cap (A - \{x\}) \neq \emptyset \text{ for all } r > 0.$$

The set of all limit points of A is called the derived set of A and is denoted by D(A).

Note : x is not a limit point of A iff there exists an open ball $B(x,r)$ such that $B(x,r) \cap (A - \{x\}) = \emptyset$.

Example 1 : Consider R with usual metric.

(a) Let A=[0,1).

Any open ball with centre 0 is of the form (-r,r) which contains a point of [0,1) other than 0.

Hence 0 is a limit point of [0,1).

Similarly 1 is a limit point of [0,1).

2 is not a limit point of A, since $\left(2 - \frac{1}{2}, 2 + \frac{1}{2}\right) \cap [0,1) = \left(\frac{3}{2}, \frac{5}{2}\right) \cap [0,1) = \emptyset$.

In this case all points of [0,1] are limit points of [0,1) and no other point is a limit point.

Hence D([0,1)) = [0,1].

(b) Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\dots\dots\dots, \frac{1}{n}, \dots..\right\}$. Here 0 is a limit point of A.

For, consider any open ball $(-r, r)$ with centre 0.

Choose a positive integer n such that $\frac{1}{n} < r$.

Then $\frac{1}{n} \in (-r, r)$.

$\therefore$ $(-r, r)$ contains a point of A which is different from 0.

$\therefore$ 0 is a limit point of A.

1 is not a limit point of A since, $\left(1 - \frac{1}{4}, 1 + \frac{1}{4}\right) \cap (A - \{1\})$

$$= \left(\frac{3}{4}, \frac{5}{4}\right) \cap \left\{\frac{1}{2}, \frac{1}{3}, \ldots \ldots \ldots, \frac{1}{n}, \ldots \ldots\right\} = \emptyset.$$

In fact any point except zero is not a limit point of A.

$\therefore D(A) = \{0\}$.

(c) Z has no limit point.

Proof : Let x be any real number.

If x is an integer, then $B\left(x, \frac{1}{2}\right) = \left(x - \frac{1}{2}, x + \frac{1}{2}\right)$ does not contain any integer other than x. Hence x is not a limit point of Z.

If x is not an integer, let n be the integer which is closest to x.

Choose r such that $0 < r < |x - n|$.

Then $B(x, r) = (x - r, x + r)$ contains no integer.

Hence x is not a limit point of Z.

Since x is arbitrary, Z has no limit point.

$\therefore D(Z) = \emptyset$.

(d) Consider Q. Any real number x is a limit point of Q, since the interval $(x - r, x + r)$ contains infinite number of rational numbers.

$$\therefore D(Q) = R.$$

Example 2 : In $R \times R$ with usual metric, $D(Q \times Q) = R \times R$.

The proof is similar to example (d) of 1.

Example 3 : Let (M,d) be a discrete metric space.

Let $A \subseteq M$. Let $x \in M$.

Then $B\left(x, \frac{1}{2}\right) \cap (A - \{x\}) = \{x\} \cap (A - \{x\}) = \emptyset$.

$\therefore$ x is not a limit point of A.

Since $x \in M$ is arbitrary, A has no limit point.

$$\therefore D(A) = \emptyset.$$

Thus any subset of a discrete metric space has no limit point.

Example 4 : Consider C with usual metric.

Let $A = \{z/|z| < 1\}$

Then $D(A) = \{z/|z| \leq 1\}$.

Theorem 2.8 : Let (M,d) be a metric space. Let $A \subseteq M$. Then x is a limit point of A iff each open ball with centre x contains an infinite number of points of A.

Proof : Let x be a limit point of A.

To prove each open ball with centre x contains an infinite number of points of A.

Suppose an open ball $B(x,r)$ contains only a finite number of points of A.

Let $B(x,r) \cap (A - \{x\}) = \{x_1, x_2, \ldots \ldots \ldots \ldots, x_n\}$

Let $r_1 = \min\{d(x, x_i)/i = 1,2, \ldots \ldots \ldots., n\}$

Since $x \neq x_i, d(x, x_i) > 0$ for all i=1,2,.......,n and hence $r_1 > 0$.

Also, $B(x,r) \cap (A - \{x\}) = \emptyset$

$\therefore x$ is not a limit point of A which is a contradiction.

Hence every open ball with centre x contains infinite number of points of A.

Conversely, if each open ball with centre x contains an infinite number of points of A then obviously x is a limit point of A.

Corollary : Any finite subset of a metric space has no limit point.

Proof : Let A be a finite subset of M.

To prove A has no limit point.

Suppose A has a limit point say x.

Then each open ball with centre x contains infinite number of points of A.

This is a contradiction since A is finite.

Hence, A has no limit point.

Theorem 2.9 : Let M be a metric space and $A \subseteq M$. Then $\bar{A} = A \cup D(A)$.

Proof : Let $x \in A \cup D(A)$. We shall prove that $x \in \bar{A}$.

Suppose $x \notin \bar{A}$

$\therefore x \in M - \bar{A}$ and since $\bar{A}$ is closed, $M - \bar{A}$ is open.

$\therefore B(x,r) \cap \bar{A} = \emptyset$

$\therefore B(x,r) \cap A = \emptyset$ $[since\ A \subseteq \bar{A}]$

$\therefore x \notin A \cup D(A)$ which is a contradiction.

$\therefore x \in \bar{A}$.

$\therefore A \cup D(A) \subseteq \bar{A}$. .... (1)

Now, let $x \in \bar{A}$.

To prove $x \in A \cup D(A)$.

If $x \in A$ then clearly $x \in A \cup D(A)$.

Suppose $x \notin A$. We claim that $x \in D(A)$.

Suppose $x \notin D(A)$. Then there exists an open ball $B(x,r)$ such that $B(x,r) \cap A = \emptyset$.

$\therefore B(x,r)^c \supseteq A$.

Since $B(x,r)$ is open, $B(x,r)^c$ is closed.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subseteq B(x,r)^c$.

But $x \in \bar{A}$ and $x \notin B(x,r)^c$ which is a contradiction.

Hence, $x \in D(A)$.

$\therefore x \in A \cup D(A)$.

$\therefore \bar{A} \subseteq A \cup D(A)$. .... (2)

From (1) and (2) $\bar{A} = A \cup D(A)$.

Corollary 1 : A is closed iff A contains all its limit points. i.e., A is closed iff $D(A) \subseteq A$.

Proof : A is closed $\Leftrightarrow A = \bar{A}$ [By theorem 2.6]

$\therefore x \in A$ or $x \in D(A)$.

If $x \in A$ then $x \in B(x,r) \cap A$.

If $x \in D(A)$ then $B(x,r) \cap A \neq \emptyset$ for all $r > 0$.

Hence in both cases $B(x,r) \cap A \neq \emptyset$ for all $r > 0$.

Conversely, suppose $B(x,r) \cap A \neq \emptyset$ for all $r > 0$.

We have to prove that $x \in \bar{A}$.

If $x \in A$ trivially $x \in \bar{A}$.

Let $x \notin A$. Then $A - \{x\} = A$.

$\therefore B(x,r) \cap (A - \{x\}) \neq \emptyset$.

$\therefore x \in D(A)$.

$\therefore x \in \bar{A}$.

Corollary 3 : $x \in \bar{A} \Leftrightarrow G \cap A \neq \emptyset$ for every open set G containing x.

Proof : Let $x \in \bar{A}$.

Let G be an open set containing x. Then there exists $r > 0$ such that $B(x,r) \subseteq G$.

Also, since $x \in \bar{A}$, $B(x,r) \cap A \neq \emptyset$.

$\therefore G \cap A \neq \emptyset$.

Conversely, suppose $G \cap A \neq \emptyset$ for every open set G containing x.

Since $B(x,r)$ is an open set containing x, we have $B(x,r) \cap A \neq \emptyset$.

$\therefore x \in \bar{A}$.

Example 1 : Consider R with ususal metric.

(a) Let $A = [0,1)$.

Then $\bar{A} = A \cup D(A)$.

$= [0,1) \cup [0,1]$

$= [0,1]$.

(b) Let $A = \left\{1, \frac{1}{2}, \dots \dots .., \frac{1}{n}, \dots ..\right\}$

Then $\bar{A} = A \cup D(A)$.

$=\left\{1, \frac{1}{2}, \dots \dots .., \frac{1}{n}, \dots ..\right\} \cup \{0\}$.

(c) $\bar{Z} = Z \cup D(Z)$

$= Z \cup \emptyset = Z$.

$\therefore$ Z is closed.

(d) $\bar{Q} = Q \cup D(Q)$

$= Q \cup R$

$= R$.

$\therefore$ Q is not closed.

Example 2 : In $R \times R$ with usual metric.

$\overline{Q \times Q} = (Q \times Q) \cup D(Q \times Q)$

$= (Q \times Q) \cup (R \times R)$

$= R \times R$.

$\therefore Q \times Q$ is not closed.

SOLVED PROBLEM

Problem 1 : Prove that for any subset A of a metric space, $d(A) = d(\bar{A})$ where $d(A)$ is the diameter of A.

Solution : We have $A \subseteq \bar{A}$.

$\therefore d(A) \leq d(\bar{A})$. .... (1)

Now, let $\varepsilon > 0$ be given. We claim that $d(\bar{A}) \leq d(A) + \varepsilon$.

Let $x, y \in \bar{A}$.

$\therefore B\left(x, \frac{1}{2}\varepsilon\right) \cap A \neq \emptyset$ and $B\left(y, \frac{1}{2}\varepsilon\right) \cap A \neq \emptyset$ [By Corollary 2]

Let $x_1 \in B\left(x, \frac{1}{2}\varepsilon\right) \cap A$ and $x_2 \in B\left(y, \frac{1}{2}\varepsilon\right) \cap A$

$\therefore x_1 \in B\left(x, \frac{1}{2}\varepsilon\right)$ and $x_2 \in B\left(y, \frac{1}{2}\varepsilon\right)$.

$\therefore d(x, x_1) < \frac{\varepsilon}{2}$ and $d(y, x_2) < \frac{\varepsilon}{2}$ .... (2)

Also, $x_1 \in A$ and $x_2 \in A \Rightarrow d(x_1, x_2) \leq d(A)$ .... (3)

Now, $d(x, y) \leq d(x, x_1) + d(x_1, x_2) + d(x_2, y)$

$< \frac{1}{2}\varepsilon + d(A) + \frac{1}{2}\varepsilon$ [by (2)& (3)]

$= d(A) + \varepsilon$.

Thus $d(x, y) \leq d(A) + \varepsilon$.

$\therefore l.u.b\ \{d(x, y)/x, y \in \bar{A}\} \leq d(A) + \varepsilon$.

i.e., $d(\bar{A}) \leq d(A) + \varepsilon$.

Now, since ε is arbitrary, we have $d(\bar{A}) \leq d(A)$ .... (4)

By (1) and (4) we get $d(A) = d(\bar{A})$.

DENSE SETS

Definition : A subset A of a metric space M is said to be dense in M or everywhere dense if $\bar{A} = M$.

Definition : A metric space M is said to be separable if there exists a countable dense subset in M.

Example 1 : Let M be a metric space. Trivially, M is dense in M.

Hence any countable metric space is separable.

Example 2 : In R with usual metric Q is dense in R since $\bar{Q} = R$.

Further Q is countable.

Hence R is separable.

Example 3 : Let M be a discrete metric space.

Let $A \subset M$ and since $A \neq M$.

Since A is closed, $\bar{A} = A$.

$\therefore$ A is not dense.

Hence, any countable discrete metric space is not separable.

Example 4 : In $R \times R$ with usual metric $Q \times Q$ is a dense set since $\overline{Q \times Q} = R \times R$.

Also Q is countable and hence $Q \times Q$ is countable.

$\therefore R \times R$ is separable.

Theorem 2.10 : Let M be a metric space and $A \subseteq M$. Then the following are equivalent.

(i) A is dense in M.

(ii) The only closed set which contains A is M.

(iii) The only open set disjoint from A is $\emptyset$.

(iv) A intersects every nonempty open set.

(v) A intersects every open ball.

Proof : (i)⇒(ii)

Suppose A is dense in M.

Then $\bar{A} = M$. .... (1)

Now, let $F \subseteq M$ be any closed set containing A.

Since, $\bar{A}\ is$ th smallest closed set containing A, we have $\bar{A} \subseteq F$.

Hence, $M \subseteq F$. [by (1)]

Hence, $M = F$.

$\therefore$ The only closed set which contains A is M.

(ii)⇒(iii)

Suppose (iii) is not true.

Then there exists a non-empty open set B such that $B \cap A = \emptyset$.

$\therefore B^c$ is a closed set and $B^c \supseteq A$.

Further, since $B \neq \emptyset$ we have $B^c \neq M$ which is a contradiction to (ii).

Hence, (ii)⇒(iii)

(iii)⇒(iv) is obvious.

(iv)⇒(v), since every open ball is an open set we get the result.

(v)⇒(i)

Let $x \in M$.

Suppose every open ball $B(x, r)$ intersects A.

Then by Corollary 2 of Theorem 2.9, $x \in \bar{A}$.

$\therefore M \subseteq \bar{A}$.

But trivially $\bar{A} \subseteq M$.

$\therefore \bar{A} = M$. Hence, A is dense in M.

SOLVED PROBLEM

Problem 1 : Give an example of a set E such that both E and E^c are dense in R.

Solution : Let $E = Q$.

Since any open ball $B(x,r) = (x-r, x+r)$ contains both rational and irrational numbers Q and Q^c.

Hence Q and Q^c are dense in R.

COMPLETE METRIC SPACE

Definition : Let (M,d) be a metric space. Let $(x_n) = x_1, x_2, \ldots \ldots., x_n, \ldots \ldots$ be a sequence of points in M. Let $x \in M$. We say that (x_n) converges to x if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$. Also x is called a limit of (x_n).

If (x_n) converges to x we write $\lim_{n\to\infty} x_n = x \; or \; (x_n) \to x$.

Note 1 :$(x_n) \to x$ iff for each open ball $B(x,\varepsilon)$ with centre x there exists a positive integer n_0 such that $x_n \in B(x,\varepsilon)$ for all $n \geq n_0$.

Thus the open ball $B(x,\varepsilon)$ contains all but a finite number of terms of the sequence.

Note 2 :$(x_n) \to x$ iff the sequence of real numbers $\big(d(x_n, x)\big) \to 0$.

Theorem 2.11 : For a convergent sequence (x_n) the limit is unique.

Proof : Suppose $(x_n) \to x$ and $(y_n) \to y$.

Let $\varepsilon > 0$ be given.

Since, $(x_n) \to x$, there exists positive integern_1 such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq n_1$.

Also, since $(y_n) \to y$, there exists positive integer n_2 such that $d(y_n, y) < \frac{\varepsilon}{2}$ for all $n \geq n_2$.

Let m be a positive integer such that $m \geq n_1, n_2$.

Then $d(x,y) \leq d(x, x_m) + d(x_m, y)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

$\therefore d(x,y) < \varepsilon.$

Since $\varepsilon > 0$ is arbitrary, $d(x,y) = 0$.

$\therefore x = y.$

Note : In view of the above theorem if $(x_n) \to x$, then x is called of the limit of the sequence (x_n).

Theorem 2.12: Let M be a metric space and $A \subseteq M$. Then

(i) $x \in \bar{A}$ iff there exists a sequence (x_n) in A such that $(x_n) \to x$.

(ii) x is a limit point of A iff there exists a sequence (x_n) of distinct points in A such that $(x_n) \to x$.

Proof : Let $x \in \bar{A}$. Then $x \in A \cup D(A)$.

$\therefore x \in A \text{ or } x \in D(A)$.

If $x \in A$, then the sequence $x, x, x, \ldots..$ is a sequence in A converging to x.

If $x \in D(A)$ then the open ball $B\left(x, \frac{1}{n}\right)$ contains infinite number of points of A.

$\therefore$ We can choose $x_n \in B\left(x, \frac{1}{n}\right) \cap A$ such that $x_n \neq x_1, x_2, \ldots \ldots., x_{n-1}$ for each n.

$\therefore (x_n)$ is a sequence of distinct points in A.

Also, $d(x_n, x) < \frac{1}{n}$ for all n.

$\therefore \lim_{n\to\infty} d(x_n, x) = 0$.

$\therefore (x_n) \to x$.

Conversely, suppose there exists a sequence (x_n)in A such that $(x_n) \to x$.

Then for any $r > 0$ there exists a positive integer n_0 such that $d(x_n, x) < r$ for all $n \geq n_0$.

$\therefore x_n \in B(x, r) for\ all\ n \geq n_0$.

$\therefore B(x, r) \cap A \neq \emptyset$.

Hence, $x \in \bar{A}$.

Further, if (x_n) is a sequence of distinct points, $B(x, r) \cap A$ is infinite.

$\therefore x \in D(A)$.

Hence, x is a limit point of A.

Definition : Let (M,d) be a metric space. Let (x_n) be a sequence of points in M. (x_n) is said to be a Cauchy sequence in M if given $\varepsilon > 0$ there exists a positive integer n_0 such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$.

Theorem 2.13 : Let (M,d) be a metric space. Then any convergent sequence in M is a Cauchy sequence.

Proof : Let (x_n) be a convergent sequence in M converging to $x \in M$.

Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq n_0$.

$\therefore d(x_m, x_n) \leq d(x_m, x) + d(x, x_n)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ for all $m, n \geq n_0$.

$= \varepsilon$ for all $m, n \geq n_0$.

Thus, $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$.

$\therefore$ (x_n) is a Cauchy sequence.

Note : The converse of the above theorem is not true. i.e., any Cauchy sequence need not be a convergence sequence.

For example, consider the metric space (0,1] with usual metric.

$\left(\frac{1}{n}\right)$ is a Cauchy sequence in (0,1].

But this sequence does not converge to any point in (0,1].

Definition : A metric space M is said to be complete if every Cauchy sequence in M converges to a point in M.

Example 1: R with usual metric is complete. This is a fundamental fact to elementary analysis.

Note : The metric space (0,1] with usual metric is not complete.

Example 2 : C with usual metric is complete.

Proof : Let (z_n) be a Cauchy sequence in C.

Let $z_n = x_n + iy_n$ where $x_n, y_n \in R$.

We claim that (x_n)& (y_n) are Cauchy sequences in R.

Let $\varepsilon > 0$ be given.

Since (z_n) is a Cauchy sequence, there exists a positive integer $n_0$such that $|z_n - z_m| < \varepsilon$ for all $n, m \geq n_0$.

Now, $|x_n - x_m| \leq |z_n - z_m|$ and $|y_n - y_m| \leq |z_n - z_m|$

Hence, $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_0$ and $|y_n - y_m| < \varepsilon$ for all $n, m \geq n_0$.

$\therefore$ (x_n) and (y_n) are Cauchy sequences in R.

Since R is complete, there exists $x, y \in R$ such that $(x_n) \to x$ and $(y_n) \to y$.

Let $z = x + iy$.

We claim that $(z_n) \to z$.

We have $|z_n - z| = |(x_n + iy_n) - (x + iy)|$

$= |(x_n - x) + i(y_n - y)|$

$\leq |x_n - x| + |y_n - y|$ …………..(1)

Now, let $\varepsilon > 0$ be given.

Since, $(x_n) \to x$ and $(y_n) \to y$ there exists a positive integer n_1 and n_2 such that $|x_n - x| < \frac{\varepsilon}{2}$ $for\ all\ n \geq n_1\ and\ |y_n - y| < \frac{\varepsilon}{2}$ $for\ all\ n \geq n_2$.

Let $n_3 = \max\{n_1, n_2\}$.

From (1) we get $|z_n - z| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ for all $n \geq n_3$.

$\therefore (z_n) \to z$.

$\therefore z$ is complete.

Example 3 : Any discrete metric space is complete.

Proof : Let (M,d) be a discrete metric space.

Let (x_n) be a Cauchy sequence in M.

Then there exists a positive integer n_0 such that $d(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$.

Since d is the discrete metric, distance between any two points is either 0 or 1.

$\therefore\ d(x_n, x_m) = 0$ for all $n, m \geq n_0$.

$\therefore x_n = x_{n_0} = x\ (say)$ for all $n \geq n_0$.

$\therefore d(x_n, x) = 0$ for all $n \geq n_0$.

$\therefore (x_n) \to x$.

Hence M is complete.

Example 4 :R^n with usual metric is complete.

Proof : Let (x_p) be a Cauchy sequence in R^n.

Let $(x_p) = (x_{p_1}, x_{p_2}, \dots \dots \dots \dots \dots x_{p_n})$.

Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geq n_0$.

$\therefore \left[\sum_{k=1}^{n} (x_{p_k} - x_{q_k})^2\right]^{1/2} < \varepsilon$ for all $p, q \geq n_0$.

$\therefore \sum_{k=1}^{n} (x_{p_k} - x_{q_k})^2 < \varepsilon^2$for all $p, q \geq n_0$.

$\therefore$ For each k=1,2, ….., n we have $\left|x_{p_k} - x_{q_k}\right| < \varepsilon$ for all $p, q \geq n_0$.

$\therefore (x_{p_k})$is a Cauchy sequences in R for each k=1,2,……..,n.

Since R is complete, there exists $y_k \in R$ such that $(x_{p_k}) \to y_k$.

Let $y = (y_1, y_2, \dots \dots \dots \dots \dots y_n)$.

We claim that $(x_p) \to y$.

Since $(x_{p_k}) \to y_k$ there exists a positive integer m_k such that $|x_{p_k} - y_k| < \frac{\varepsilon}{\sqrt{n}}$ for all $p \geq m_k$.

Let $m_0 = \max\{m_1, m_2, \dots \dots . m_n\}$.

Then $d(x_p, y) = \left[\sum_{k=1}^{n} (x_{p_k} - y_k)^2\right]^{1/2}$

$$< \left[n\left(\frac{\varepsilon}{\sqrt{n}}\right)^2\right]^{1/2} \text{ for all } p \geq m_0.$$

$$= \varepsilon \text{ for all } p \geq m_0.$$

$\therefore d(x_p, y) < \varepsilon$for all $p \geq m_0$.

$$\therefore (x_p) \to y.$$

Hence, R^n is complete.

Example 5 :l_2 is complete.

Proof : Let (x_p) be a Cauchy sequence in l_2.

Let $(x_p) = (x_{p_1}, x_{p_2}, \dots \dots \dots \dots \dots x_{p_n})$.

Let $\varepsilon > 0$ be given.

Then there exists a positive integer n_0 such that $d(x_p, x_q) < \varepsilon$ for all $p, q \geq n_0$.

$\therefore \left[\sum_{n=1}^{\infty} (x_{p_n} - x_{q_n})^2\right]^{1/2} < \varepsilon$ for all $p, q \geq n_0$.

$\therefore \sum_{n=1}^{\infty} (x_{p_n} - x_{q_n})^2 < \varepsilon^2$for all $p, q \geq n_0$. ………..(1)

$\therefore$ For each n=1,2, ….., …. we have $|x_{p_n} - x_{q_n}| < \varepsilon$ for all $p, q \geq n_0$.

$\therefore$ (x_{p_n}) is a Cauchy sequences in R for each n.

Since R is complete, there exists $y_k \in R$ such that $\left(x_{p_n}\right) \to y_n$. ……..(2)

Let $y = (y_1, y_2, \dots \dots \dots \dots \dots y_n, \dots \dots .)$.

We claim that $y \in l_2$ and $\left(x_p\right) \to y$.

For any fixed positive integer m, we have $\sum_{n=1}^{m}(x_{p_n} - x_{q_n})^2 < \varepsilon^2$for all $p, q \geq n_0$.

[using (1)]

Fixing q and allowing $p \to \infty$ in this finite sum we get

$\sum_{n=1}^{m}(y_n - x_{q_n})^2 < \varepsilon^2$for all $q \geq n_0$.

Since this is true for every positive integer m$\sum_{n=1}^{\infty}(y_n - x_{q_n})^2 < \varepsilon^2$for all $q \geq n_0$.

……………. (3)

Now, $[\sum_{n=1}^{\infty}|y_n|^2]^{\frac{1}{2}} = \left[\sum_{n=1}^{\infty}\left|y_n - x_{q_n} + x_{q_n}\right|^2\right]^{\frac{1}{2}}$

$=\left[\sum_{n=1}^{\infty}\left|y_n - x_{q_n}\right|^2\right]^{\frac{1}{2}} + \left[\sum_{n=1}^{\infty}\left|x_{q_n}\right|^2\right]^{\frac{1}{2}}$ [By Minkowski's inequality]

$\leq \varepsilon + \left[\sum_{n=1}^{\infty}\left|x_{q_n}\right|^2\right]^{\frac{1}{2}}$ for all $q \geq n_0$ (by (3)

Since $x_q \in l_2$ we have $\left[\sum_{n=1}^{\infty}\left|x_{q_n}\right|^2\right]^{\frac{1}{2}}$ converges.

$\therefore$ $[\sum_{n=1}^{\infty}|y_n|^2]^{\frac{1}{2}}$converges.

$$\therefore y \in l_2.$$

Also (3) gives $d(y, x_q) \leq \varepsilon$ for all $q \geq n_0$.

$$\therefore \left(x_p\right) \to y.$$

$\therefore$ l_2 is complete.

Note : A subspace of a complete metric space need not be complete.

For example, R with usual metric is complete. But the subspace (0,1] is not complete.

Theorem 2.14 : A subset A of a complete metric space M is complete iff A is closed.

Proof : Suppose A is complete.

To prove A is closed.

We shall prove that A contains all its limit points.

Let x be a limit point of A.

Then by theorem, there exists a sequence (x_n)in A such that $(x_n) \to x$.

Since A is complete, $x \in A$.

$\therefore A$ contains all its limit points.

Hence A is closed.

Conversely, let A be a closed subset of M.

To prove A is complete.

Let (x_n) be a Cauchy sequence in A.

Then (x_n) is a Cauchy sequence in M also and since M is complete there exists $x \in M$ such that $(x_n) \to x$.

Thus (x_n) is a sequence in A converging to x.

$\therefore x \in \bar{A}$ (by theorem)

Now, since A is closed, $A = \bar{A}$.

$\therefore x \in A$.

Thus every Cauchy sequence (x_n) in A converges to a point in A.

Hence A is complete.

Note 1 : [0,1] with usual metric is complete since it is a closed subset of the complete metric space R.

Note 2 : Consider Q. Since $\bar{Q} = R$, Q is not a closed subset of R.

Hence Q is not complete.

Solved problems

Problem 1 : Let A, B be subsets of R. Prove that $\overline{A \times B} = \bar{A} \times \bar{B}$.

Solution : Let $(x, y) \in \overline{A \times B}$

$\therefore$ There exists a sequence $((x_n, y_n)) \in A \times B$ such that $\big((x_n, y_n)\big) \to (x, y)$.

$\therefore$ $(x_n) \to x$ and $(y_n) \to y$.

Also (x_n) is a sequence in A and (y_n) is a sequence in B.

$\therefore$ $x \in \bar{A}$ and $y \in \bar{B}$.

$\therefore$ $(x, y) \in \bar{A} \times \bar{B}$.

$\therefore$ $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$. ……………(1)

Now, let $(x, y) \in \bar{A} \times \bar{B}$.

$\therefore$ $x \in \bar{A}$ and $y \in \bar{B}$.

$\therefore$ There exists a sequence (x_n) in A and a sequence (y_n) in B such that $(x_n) \to x$ and $(y_n) \to y$.

$\therefore$ $\big((x_n, y_n)\big) is\ a\ sequence\ in\ A \times B$ such that $\big((x_n, y_n)\big) \to (x, y)$.

Hence $(x, y) \in \overline{A \times B}$

$\therefore$ $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$. ……………(2)

From (1) & (2) $\overline{A \times B} = \bar{A} \times \bar{B}$.

Problem 2 :If A and B are closed subsets of R. Prove that $A \times B$ is a closed subset of $R \times R$.

Solution : Since A and B are closed sets we have $A = \bar{A}$ and $B = \bar{B}$.

Now, $\overline{A \times B} = \bar{A} \times \bar{B}$ $[By\ problem\ 1]$

$= A \times B$.

$\therefore$ $A \times B$ is a closed set.

Theorem 2.15 (Cantor's Intersection Theorem)

Statement : Let (M,d) be a metric space. M is complete iff for every sequence (F_n) of non-empty closed subsets of M such that $F_1 \supseteq F_2 \supseteq \cdots \ldots \ldots \ldots. \supseteq F_n \supseteq \cdots \ldots$ and $(d(F_n)) \to 0$, then $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

Proof :Let (M,d) be a complete metric space.

Let (F_n)be a sequence of non-empty closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \cdots \ldots \ldots \ldots . \supseteq F_n \supseteq \cdots \ldots$ ………. (1)

and $(d(F_n)) \to 0$ ………. (2)

We claim that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

For each positive integer n, choose a point $x_n \in F_n$.

By (1), x_n, x_{n+1}, x_{n+2} … … … … …all lie in F_n.

i.e., $x_m \in F_n$ for all $m \geq n$. ………….. (3)

Since, $(d(F_n)) \to 0$, given $\varepsilon > 0$, there exists a positive integer n_0, such that $d(F_n) < \varepsilon$ for all $n \geq n_0$.

In particular $d(F_{n_0}) < \varepsilon$. …………… (4)

$\therefore d(x, y) < \varepsilon$ for all $x, y \in F_n$.

Now, $x_m \in F_{n_0}$ for all $m \geq n_0$. $[by\ (3)]$

$\therefore m, n \geq n_0 \Rightarrow x_m, x_n \in F_{n_0}$.

$\Rightarrow d(x_m, x_n) < \varepsilon$ [By (4)]

$\therefore (x_n)$ is a Cauchy sequence in M.

Since M is complete there exists a point $x \in M$ such that $(x_n) \to x$.

We claim that $x \in \bigcap_{n=1}^{\infty} F_n$.

Now, for any positive integer n, x_n, x_{n+1}, x_{n+2} … … … … … is a sequence in $F_n$and this sequence converges to x.

$\therefore x \in \overline{F_n}$.

But $\overline{F_n}$ is closed and hence $\overline{F_n} = F_n$.

$\therefore x \in F_n$.

$$\therefore x \in \bigcap_{n=1}^{\infty} F_n.$$

Hence, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Conversely, assume that for every sequence (F_n) of non-empty closed subsets of M such that $F_1 \supseteq F_2 \supseteq \cdots \ldots \ldots \ldots . \supseteq F_n \supseteq \cdots \ldots$ and $(d(F_n)) \to 0,$ then $\bigcap_{n=1}^{\infty} F_n$ is nonempty.

To prove M is complete.

Let (x_n) be a Cauchy sequence in M.

Let $\quad F_1 = \{x_1, x_2, x_3, \ldots \ldots \ldots ., x_n, \ldots ..\}$

$F_2 = \{x_2, x_3, \ldots \ldots \ldots ., x_n, \ldots ..\}$

…………………………………

…………………………………

$F_n = \{x_n, x_{n+1}, \ldots \ldots \ldots \ldots \ldots ..\}$

……………………………….

Clearly, $F_1 \supseteq F_2 \supseteq \cdots \ldots \ldots \ldots . \supseteq F_n \supseteq \cdots \ldots$

$\therefore \overline{F_1} \supseteq \overline{F_2} \supseteq \cdots \ldots \ldots \ldots . \supseteq \overline{F_n} \supseteq \cdots \ldots$

$\therefore (\overline{F_n})$ is a decreasing sequence of closed sets.

Now, since (x_n) is a Cauchy sequence, given $\varepsilon > 0$ there exists a positive integer n_0, such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq n_0$.

$\therefore$For any integer $n \geq n_0$, the distance between any two points of F_n is less than ε.

$\therefore d(F_n) < \varepsilon$ for all $n \geq n_0$.

But $d(F_n) = d(\overline{F_n})$.

$\therefore d(\overline{F_n}) < \varepsilon$ for all $n \geq n_0$. ……………… (5)

$$\therefore d(\overline{F_n}) \to 0.$$

Hence, $\bigcap_{n=1}^{\infty} \overline{F_n} \neq \emptyset$.

Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$.

Then x and $x_n \in \overline{F_n}$.

$$\therefore d(x_n, x) \leq \overline{F_n}.$$

$$< \varepsilon \text{ for all } n \geq n_0. \quad \text{[From (5)]}$$

$$\therefore (x_n) \to x.$$

$\therefore$ M is complete.

Note 1 : In the above theorem $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

For, suppose $\bigcap_{n=1}^{\infty} F_n$ contains two distinct points x and y.

Then $d(F_n) \geq d(x, y)$ for all n.

$\therefore d(F_n)$ does not tend to zero which is a contradiction.

$\therefore \bigcap_{n=1}^{\infty} F_n$ contains exactly one point.

Note 2 :In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if each F_n is not closed.

For example, consider $F_n = \left(0, \frac{1}{n}\right)$ $in\ R$.

Clearly, $F_1 \supseteq F_2 \supseteq \cdots \ldots \ldots \ldots . \supseteq F_n \supseteq \cdots \ldots$ and $(d(F_n)) \to 0$ as $n \to \infty$.

But, $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Note 3 : In the above theorem $\bigcap_{n=1}^{\infty} F_n$ may be empty if the hypothesis $(d(F_n)) \to 0$ is omitted.

For example, consider $F_n = [n, \infty)$ in R.

Clearly (F_n) is a sequence of closed sets and $F_1 \supseteq F_2 \supseteq \cdots \ldots \ldots \ldots . \supseteq F_n \supseteq \cdots \ldots$

Also, $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Here, $d(F_n) = \infty$ for all n and hence the hypothesis $(d(F_n)) \to 0$ is not true.

BAIRE'S CATEGORY THEOREM

Definition : A subset A of a metric space M is said to be nowhere dense in M if $Int\bar{A} = \emptyset$.

Definition : A subset A of a metric space M is said to be of first category in M if A can be expressed as a countable union of nowhere dense sets.

A set which is not of first category is said to be of second category.

Note : If A is of first category then $A = \bigcup_{n=1}^{\infty} E_n$ where E_n is nowhere dense subsets in M.

Example 1 : In R with usual metric $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots \dots \dots, \frac{1}{n}, \dots \dots \dots \right\}$ is nowhere dense.

For, $\bar{A} = A \cup D(A) = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots \dots \dots, \frac{1}{n}, \dots \dots \dots \right\}$

Clearly $Int\bar{A} = \emptyset$.

Example 2 : In any discrete metric space M, any non-empty subset A is not nowhere dense.

For, in a discrete metric space every subset is both open and closed.

$\therefore \bar{A} = Int\bar{A} = IntA = A$.

$\therefore Int\bar{A} \neq \emptyset$.

$\therefore$ A is not nowhere dense.

Example 3 : In R with usual metric any finite subset A is nowhere dense.

For, let A be any finite subset of R.

Then A is closed and hence $A = \bar{A}$.

Also since A is finite, no point of A is an interior point of A/

$\therefore Int\bar{A} = IntA = \emptyset$.

$\therefore$ A is nowhere dense.

Example 4 : Consider R with usual metric. Any singleton set {x} is nowhere dense.

$\therefore$ Any countable subset of R being a countable union of singleton sets is of first category.

In particular Q is of first category.

Note : If A and B are sets of first category in a metric space M then $A \cup B$ is also of first category.

For, since A and B are of first category in M we have $A = \bigcup_{n=1}^{\infty} E_n$ and $B = \bigcup_{n=1}^{\infty} H_n$ where E_n and H_n are nowhere dense subsets in M.

$\therefore A \cup B$ is a countable union of nowhere dense subsets of M.

Hence $A \cup B$ is of first category.

Theorem 2.16 (Equivalent characterisations of nowhere dense sets

Let M be a metric space and $A \subseteq M$. Then the following are equivalent.

(i) A is nowhere dense in M.

(ii) $\bar{A}$ dos not contain any non-empty open set.

(iii) Each non-empty open set has a non-empty open subset disjoint from $\bar{A}$.

(iv) Each non-empty open set has a non-empty open subset disjoint from A.

(v) Each non-empty open set contains an open sphere disjoint from A.

Theorem 2.17 (Baire's Category Theorem)

Statement : Any complete metric space is of second category.

Proof : Let M be a complete metric space.

To prove M is of second category.

i.e. to prove M is not of first category.

Let (A_n) be a sequence of nowhere dense sets in M.

We claim that $\bigcup_{n=1}^{\infty} A_n \neq M$.

Since, M is open and A_1 is nowhere dense, there exists an open ball B_1 of radius less than 1 such that B_1 is disjoint from A_1.

Let F_1 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_1.

Now $Int\ F_1$ is open and A_2 is nowhere dense.

$\therefore IntF_1$ contains an open ball B_2 of radius less than $\frac{1}{2}$ such that B_2 is disjoint from A_2.

Let F_2 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_2.

Now $Int\ F_2$ is open and A_3 is nowhere dense.

$\therefore IntF_2$ contains an open ball B_3 of radius less than $\frac{1}{4}$ such that B_3 is disjoint from A_3.

Let F_3 denote the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_3.

Proceding like this we get a sequence of non-empty closed balls F_n such that $F_1 \supseteq F_2 \supseteq \cdots \ldots \ldots \ldots . \supseteq F_n \supseteq \cdots \ldots$ and $d(F_n) < \frac{1}{2^n}$.

Hence $\left(d(F_n)\right) \to 0$ as $n \to \infty$.

Since, M is complete, by Cantor's Intersection theorem, there exists a point x in M such that $x \in \bigcap_{n=1}^{\infty} F_n$.

Also, each F_n is disjoint from A_n.

Hence, x$\notin A_n$ for all n.

$\therefore x \notin \bigcup_{n=1}^{\infty} A_n$.

$\therefore \bigcup_{n=1}^{\infty} A_n \neq M$.

Hence M is of second category.

Corollary : R is of second category.

Proof : We know that R is a complete metric space. Hence R is of second category.

Note : The converse of the above theorem is not true.

i.e., A metric space which is of second category need not be complete.

For example, Consider M=R-Q, the space of irrational numbers.

We know that Q is of first category.

Suppose M is of first category. Then MUQ=R is also of first category which is a contradiction.

Hence, M is of second category.

Also M is not a closed subspace of R and hence M is not complete.

Exercises

I. Determine which of the following statements are true and which are false.

1. In R with discrete metric Z is a bounded set.
2. In R with usual metric Z is a bounded set.
3. In a discrete metric space every subset is bounded.
4. A subset of a metric space is bounded iff its diameter is finite.
5. A non-empty subset of a metric space is bounded iff its diameter is finite.
6. Any open ball is a non-empty set.
7. Any open ball is a bounded set.
8. In a discrete metric space M any open ball is either a singleton set or the whole space.
9. In R with usual metric [0,1) is neither open nor closed.
10. A set is closed iff its complement is open.

II. Prove that any nonempty open interval (a,b) in R is of second category.

III. Prove that a closed set A in a metric space M is nowhere dense iff A^c is everywhere dense.

IV. Prove that union of a countable number of sets which are of first category is again of first category.

UNIT III

CONTINUITY

Definition:

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. Let $f: M1 \to M2$ be a function. Let $a \in M_1$ and $l \in M_2$. The function f is said to have the limit l as $x \to a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < d_1(x,a) < \delta \Rightarrow d_2(f(x), l) < \varepsilon$. We write $\lim_{x\to a} f(x) = l$.

Let (M_1,d_1) and (M_2,d_2) be the two metric spaces. Let $a \in M_1$. A function $f: M_1 \to M_2$ is said to be continuous at a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x,a) < \delta \Rightarrow d_2(f(x),f(a)) < \varepsilon$. f is said to be continuous if it is continuous at every point of M.

Note:

1. f is continuous at a iff $\lim_{x\to a} f(x) = f(a)$.
2. The condition $d_1(x,a) < \delta \Rightarrow d_2(f(x),f(a)) < \varepsilon$ can be rewritten as
   i. $x \in B(a,\delta) \Rightarrow f(x) \in B(f(a),\varepsilon)$ *or*
   ii. $f(B(a,\delta)) \subseteq B(f(a),\varepsilon)$

Examples:

1. Let $f: M_1 \to M_2$ be given by $f(x)=a$ where $a \in M_2$ is a fixed element.

Proof:

Let $x \in M_1$ and $\varepsilon > 0$ be given

Then for any $\delta > 0$, $f(B(x,\delta)) = \{a\} \subseteq B(a,\varepsilon) = B(f(x),\varepsilon)$

$$\therefore f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$$

Since $x \in M_1$ is arbitrary, f is continuous.

2. Let (M_1,d_1) be a discrete metric and (M_2,d_2)be any metric space. Them any function $f:M_1\to M_2$ is continuous i.e, any function whose domain is a discrete metric space is continuous.

Proof:

Let $x\in M_1$ and $\varepsilon > 0$be given

Since M_1 is discrete, for any $\delta > 1, B(x,\delta) = \{x\}$

$$f(B(x,\delta)) = f(x) \subseteq B(f(x),\varepsilon)$$

Since $x\in M_1$ is arbitrary, f is continuous.

Theorem 3.1

Let (M_1,d_1) and (M_2,d_2) be the two metric spaces. Let $a\in M_1$. A function $f:M_1\to M_2$ is continuous at a iff $(x_n)\to a \Rightarrow (f(x_n))\to f(a)$

proof:

Assume that f is continuous at a.

Let (x_n) be a sequence in M_1 such that $(x_n)\to a$

We claim that $(f(x_n))\to f(a)$

Let $\varepsilon > 0$be given

By the definition of continuity, there exists $\delta > 0$ such that

$d_1(x,a)<\delta \Rightarrow d_1(f(x),f(a))< \varepsilon$ -------------------(1)

Also $(x_n)\to a$ there exists a positive integer n_o such that $d_1(x_n,a)<\delta$ for all $n>n_o$

$\therefore d_2(f(x_n),f(a))< \varepsilon$ for all $n>n_o$ [from(1)]

$\therefore (f(x_n))\to f(a)$

Conversly, assume that $(x_n)\to a \Rightarrow (f(x_n))\to f(a)$

We claim that f is continuous at a.

Suppose f is not continuous at a.

Then there exists $\varepsilon > 0$ such that for all$\delta > 0$

$(f(B(a,\delta)) \not\subset B(f(a),\varepsilon)$

In particular, $(f(B(a,\frac{1}{n})) \not\subset B(f(a),\varepsilon)$

Choose x_n such that $x_n \in B(a,\frac{1}{n})$ and $f(x_n) \notin B(f(a),\varepsilon)$

$x_n \in B(a,\frac{1}{n}) \Rightarrow d1(x_n,a) <,\frac{1}{n}$

$$f(x_n) \notin B(f(a),\varepsilon) \Rightarrow d2(f(x_n),f(a)) > \varepsilon$$

$\therefore (f(x_n)) \nrightarrow f(a)$, which is a contradiction to our assumption.

$\therefore f$ is continuous at a.

Corollary:

A function $f:M_1 \to M_2$ is continuous iff $(x_n) \to a \Rightarrow \big(f(x_n)\big) \to f(x)$.

Theorem 3.2

Let (M_1,d_1) and (M_2,d_2) be the two metric spaces. Let $a \in M_1$. A function $f:M_1 \to M_2$ is continuous iff $f^{-1}(G)$ is open in M_1 whenever G is open in M_2. ie, f is continuous iff inverse image of every open set is open.

Proof:

Suppose f is continuous

Let G be an open set in M_2

We claim that $f^{-1}(G)$ is open in M_1

If $f^{-1}(G)$ is empty, then it is open.

Let $f^{-1}(G) \neq \varphi$

Let $x \in f^{-1}(G)$

Hence $f(x) \in G$.

Since G is open, there exists an open ball $B(f(x),\varepsilon)$ such that $B(f(x),\varepsilon) \subseteq G$ --------(1)

Now by definition of continuity, there exists an open ball $B(x,\delta)$ such that

$f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$

$\therefore f(B(x,\delta)) \subseteq G$ [by(1)]

$\therefore B(x,\delta)) \subseteq f^{-1}(G)$

Since $x \in f^{-1}(G)$ is arbitrary, $f^{-1}(G)$ is open.

Conversly, suppose $f^{-1}(G)$ is open in M_1 whenever G is open in M_2.

We claim that, f is continuous

Let $x \in M_1$

Now $B(f(x),\varepsilon)$ is a open set in M_2

$\therefore f^{-1}(B(f(x),\varepsilon))$ is open in M_1 and $x \in f^{-1}(B(f(x),\varepsilon))$

There exists $\delta > 0$ such that $B(x,\delta) \subseteq f^{-1}(B(f(x),\varepsilon))$

$\therefore f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$

$\therefore$ f is continuous at x.

Since $x \in M_1$ is arbitrary, f is continuous.

Note:

1.If $f:M_1 \rightarrow M_2$ is continuous and G is open in M_1 then it if not necessary that f(G) is open in M_2.ie, under a continuous map the image of an open set need not be an open set.

For example:

Let M_1=R with discrete metric and M_2=R with usual metric.

Let $f:M_1 \rightarrow M_2$ be defined by f(x)=x.

Since M_1 is discrete every subset of M_1 is open.

Hence for any open subset G of M_2, $f^{-1}(G)$ is open in M_1

$\therefore$ f is continuous

Now, A={x} is open in M_1 but f(A)={x} is not open in M_2.

2.In the above example, f is continuous bijection whereas $f^{-1}:M_1\rightarrow M_2$ is not continuous.

For,{x} is an open set in M_2.

$(f^{-1})^{-1}(\{x\})=\{x\}$ which is not open in M_2

$\therefore f^{-1}$ is not continuous.

Thus if f is a continuous bijection f-1 need not be continuous.

We now give wet another characterization of continuous function in terms of closed sets.

Theorem 3.3

Let (M_1,d_1) and (M_2,d_2) be the two metric spaces. A function $f:M_1\rightarrow M_2$ is continuous iff $f^{-1}(F)$ is closed in M_1 whenever F is closed in M_2

Proof:

Suppose $f:M_1\rightarrow M_2$ is continuous

Let $F\subseteq M_2$ be closed in M_2

$\therefore F^c$ is open in M_2

$\therefore f^{-1}(F^c)$ is open in M_1

But $f^{-1}(F^c)=[f^{-1}(F)]^c$

$f^{-1}(F)$ is closed in M_1.

Conversely, We claim that f is continuous

Let G be an open set in M_2

$\therefore G^c$ is closed in M_1

$\therefore f^{-1}(G^c)$ is closed in M_1

$[f^{-1}(G)]^c$ is closed in M_1

$[f^{-1}(G)]$ is open in M_1

$\therefore[f^{-1}(G)]$ is open in M_1 , whenever f is continuous

G is open in M_2

We give one more characterization of continuous function in terms of closure of a set.

Theorem 3.4

Let (M_1,d_1) and (M_2,d_2) be the two metric spaces. Let $a\in M_1$. A function $f:M_1\rightarrow M_2$ is Continuous iff $f(\bar{A})\subseteq \overline{f(A)}$ for all $A\subseteq M_1$.

Proof:

Suppose f is continuous

Let$A\subseteq M_1$ then $f(A)\subseteq M_2$

Since f is continuous, $f^{-1}(\overline{f(A)})$ is closed in M_1

Also, $f^{-1}(\overline{f(A)})\supseteq A$ [$\because \overline{f(A)}\supseteq f(A)$]

But $\bar{A}$ is the smallest closed set containing A

$\therefore \bar{A} \subseteq f^{-1}(\overline{f(A)})$

$\therefore f(\bar{A})\subseteq \overline{f(A)}$

Conversely, let $f(\bar{A})\subseteq \overline{f(A)}$ for all $A\subseteq M_1$

To prove f is continuous,

We shall show that if F is closed set in M_2 then f^{-1} is closed in M_1

By hypothesis $f\left(\overline{f^{-1}(F)}\right) \subseteq \overline{f\, f^{-1}(F)}$

$\subseteq \bar{F}$

$= F$

Thus, $f\,(\overline{f^{-1}(F)} \subseteq F$

$(\overline{f^{-1}(F)} \subseteq f^{-1}(F)$

Also $f^{-1}(F)\subseteq(\overline{f^{-\iota}(F)}$

$\therefore f^{-1}(F)= f^{-1}(F)$

Hence $f^{-1}(F)$ is closed. Hence, f is continuous

Problem:1

Let f be a continuous real value function defined on; a metric space M. Let $A=\{x\in M: f(x)\geq 0\}$. Prove that A closed.

Solution:

$A= \{x\in M: f(x)\geq 0\}$

$= \{x\in M: f(x)\in [0,\infty)\}$

$= f^{-1}([0,\infty))$

Also $[0,\infty)$ is closed subset of R

Since f is continuous

$f^{-1}[0,\infty)$ is closed in M

$\therefore$ A is closed

Problem:2

Show that function f: R→Rdefined by $f(x)= \begin{cases} 0, & if\ x\ is\ irrational \\ 1, & if\ x\ is\ rational \end{cases}$ is not continuous by each of the following methods.

i. By the usual ε, δ method
ii. By the exhibiting a sequence (x_n) such $(x_n)\rightarrow x$ and $(f(x_n))$ does not converge to f(x)
iii. By the exhibiting an open set G such that $F^{-1}(G)$ is not open
iv. By exhibiting a closed subset F such that $f^{-1}(F)$ is not closed
v. By exhibiting a subset of A of R such that $f(\overline{A}) \not\subset \overline{f(A)}$

Solution:

i. To prove f is not continuous at x

We have to show that there exists an $\varepsilon > 0$ such that for all $\delta > 0$,

$f(B(x,\delta)) \not\subset B(f(x),\varepsilon)$

Let $\varepsilon = \frac{1}{2}$

For any $\delta > 0$, $B(x,\delta)=(x-\delta, x+\delta)$ contains both rational and irrational numbers

If x is rational, choose $y \in B(x,\delta)$ such that y is irrational and if x is irrational, choose $y \in B(x,\delta)$ such that y is rational

Then $|f(x) - f(y)|=1$ [by the definition of f]

i.e, $d(f(x),f(y))=1$

$\therefore f(y) \notin B(f(x),\frac{1}{2})$

Thus $y \in B(x,\delta)$ and $f(y) \notin B(f(x),\frac{1}{2})$

Hence f is not continuous at x

ii. Let $x \in R$

Suppose x is rational then $f(x)=1$

Let (x_n) be a sequence of irrational numbers such that $(x_n) \longrightarrow x$

Then $(f(x_n)) \longrightarrow 0$ and $f(x)=1$

$\therefore (f(x_n))$ does not converges to $f(x)$

Proof is similar if x is irrational

iii. Let $G=\left(\frac{1}{2},\frac{3}{2}\right)$

Clearly G is open in R

Now, $f^{-1}(G) = \{x \in R : f(x) \in G\}$

$= \{x \in R : f(x) \in \left(\frac{1}{2},\frac{3}{2}\right)\}$

$= Q$

But Q is not open in R

Thus $f^{-1}(G)$ is not open in R

∴ f is not continuous

iv. Choose F=$\left[\frac{1}{2},\frac{3}{2}\right]$

Then $f^{-1}(F)$=Q which is not closed in R

∴ f is continuous

v. Let A=Q. Then $\bar{A}$=R

∴$f(\bar{A})$=f(R) ={0,1} [by definition of f]

Also, f(A)=f(Q)={1}

∴ $f\overline{(A)}=\{\bar{1}\}=\{1\}$

$\overline{f(A)} \not\subset f(\bar{A})$

∴ f is not continuous

Problem:3

Let M_1, M_2, M_3 be a metric spaces. If $f:M_1\longrightarrow M_2$ and $g: M_2\longrightarrow M_3$ are continuous function. Prove that $gof: M_1\longrightarrow M_3$ is also continuous. i.e, Composition of two continuous functions is also continuous.

Solution:

Let G be open in M_3

Since g is continuous, $g^{-1}(G)$ is open in M_2

Now, since f is continuous, $f^{-1}(g^{-1}(G))$ is open in M_1

ie, $(gof)^{-1}(G)$ is open in M_1

∴ gof is continuous

Problem:4

Let M be a metric space. Let f:M$\longrightarrow R$ and g:M$\longrightarrow R$ be two continuous functions. Prove that f+g:M$\longrightarrow R$ is continuous.

Solution:

Let (x_n) be a sequence converging to x in M

Since f and g are continuous functions, $(f(x_n))\longrightarrow f(x)$ and $(g(x_n))\longrightarrow g(x)$

$\therefore (f(x_n)+g(x_n))\longrightarrow f(x)+g(x)$

ie, $((f+g)(x_n)) \longrightarrow (f+g)(x)$

$\therefore$ f+g is continuous

Problem:5

Let f, g be continuous real valued functions on a metric space M. Let A=$\{x:x\in M \ and \ f(x) < g(x)\}$. Prove that A is open.

Solution:

Since f and g are continuous real valued functions on M, f$-$g is also a continuous real valued function on M.

Now, A= $\{x: x\in M \ and \ f(x) < g(x)\}$

$= \{x:x\in M \ and \ f(x) - g(x) < 0\}$

$= \{x: x\in M \ and \ (f-g)(x) \in (-\infty, 0)\}$

$= (f-g)^{-1}\{(-\infty, 0)\}$

Now, $(-\infty, 0)$ is open in R and $f - g$ is continuous

Hence $(f-g)^{-1}\{(-\infty, 0)\}$ is open in M

$\therefore$ A is open in M

Problem:6

If f:R$\longrightarrow R$ and g:R$\longrightarrow R$ are bothe continuous functions on R and if h:$R^2\longrightarrow R^2$ is defined by h(x,y)=(f(x),g(y)). Prove that h is continuous on R^2

Solution: Let (x_n,y_n) be sequence in R^2 converging to (x,y)

We claim that $(h(x_n,y_n))$ converges to h(x,y)

Since $(x_n,y_n)\to(x,y)$ in R^2, $(x_n)\to(x)$ and $(y_n)\to(y)$ in R^2

Also f and g are continuous.

$\therefore (f(x_n))\to f(x)$ and $(g(y_n))\to g(y)$

$\therefore (f(x_n),g(y_n))\to(f(x),g(y))$

$\therefore (h(x_n,y_n))\to h(x,y)$

$\therefore$ h is continuous on R^2

Problem:7

Let (M,d) be a metric space. Let $a\in M$. Show that function $f:M\to R$ defined by $f(x)=d(x,a)$ is continuous

Solution:

Let $x\in M$

Let (x_n) be a sequence in M such that $(x_n)\to(x)$

We claim that $(f(x_n))\to f(x)$

Let $\varepsilon>0$ be given

Now, $|f(x_n)-f(x)|=|d(x_n,a)-d(x,a)|$

$\leq d(x_n,x)$

Since $(x_n)\to(x)$ there exists a positive integer n_1 such that $d(x_n,x)<\varepsilon$ for all $n\geq n_1$

$|f(x_n)-f(x)|<\varepsilon$ for all $n\geq n_1$

$(f(x_n))\to f(x)$

$\therefore$ f is continuous

Problem:8

Let f be a function from R^2 onto r defined by $f(x,y)=x$ for all $(x,y)\in R^2$. Show that f is continuous in R^2.

Solution:

Let $(x,y) \in R^2$

Let$((x_n,y_n))$ be a sequence in R converging to (x,y).

Then $(x_n) \to (x)$ and $(y_n) \to (y)$.

$\therefore (f(x_n,y_n)) = (x_n) \to (x) = f(x,y)$

$\therefore (f(x_n,y_n)) \to f(x,y)$

$\therefore$ f is continuous

Problem:9

Define f: $l_2 \to l_2$ as follows. If $S \in l_2$ is the sequence $S_1,S_2,\ldots\ldots$ Let f(S) be the sequence $0,S_1,S_2,\ldots..$ Show that f is continuous

Solution:

Let $y=(y_1,y_2,\ldots\ldots.,y_n) \in l_2$

Let (x_n) be a sequence in l_2 converging to y

Let $x_n=(x_{n1},x_{n2},\ldots..,x_{nk},\ldots)$

Then $(x_{n1}) \to y_1$, $(x_{n2}) \to y_2,\ldots\ldots.$, $(x_{nk}) \to y_k,\ldots\ldots.$

$\therefore (f(x_n)) = ((0, x_{n1},x_{n2},\ldots..,x_{nk},\ldots)) \to (0, y_1,y_2,\ldots..,y_k,\ldots) = f(y)$

$\therefore (f(x_n)) \to f(y)$

$\therefore$ f is continuous

Problem:10

Let G be an open subset of R. Prove that the characterization function on g defined by $\chi_G(x) = \begin{cases} 1, & if\ x \in G \\ 0, & if\ x \notin G \end{cases}$ is continuous at every point of G

Solution:

Let $x \in G$ so that $\chi_G(x)=1$

Let $\varepsilon > 0$ be given

Since G is open and $x \in G$, we can find a $\delta > 0$ such that $B(x,\delta) \subseteq G$

$\therefore \chi_G(B(x,\delta)) \subseteq \chi_G(G)$

$= \{1\}$

$= B(1, \varepsilon).$

Thus $\chi_G(B(x,\delta)) \subseteq B(\chi_G(x), e)$

Homeomorphism

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. A function $f:M_1 \rightarrow M_2$ is called a homeomorphism if

i. f is 1-1 and onto
ii. f is continuous
iii. f^{-1} is continuous

The metric spaces M_1 and M_2 are said to be homeomorphic if there exists a homeomorphism $f: M_1 \longrightarrow M_2$.

Definition:

A function $f:M_1 \longrightarrow M_2$ is said to an open map if f(G) is open in M_2 for every open set g in M_1. ie, f is an open map if the image of an open set in M_1 is an open set in M_2. A function $f:M_1 \longrightarrow M_2$ is said to be a closed map if f(G) is closed in M_2 for every closed set G in M_1. ie, f is a closed map if the image of a closed set in M_1 is a closed set in M_2.

Note 1 : Let $f:M_1 \longrightarrow M_2$ be a bijection, then f^{-1} is continuous iff f is an open map.

Proof:

$f^{-1}:M_1 \longrightarrow M_2$ is continuous iff $(f^{-1})^{-1}(G)$ is open in M_2 whenever G is open in M_1.

iff f(G) is open in M_2 whenever G is open in M_1.

iff G is open in M_1 whenever f(G) is open in M_2.

iff f is an open map.

$\therefore f^{-1}$ is continuous iff f is an open map.

Note 2 : Similarly, f^{-1} is continuous iff f is a closed map

Note 3 : Let f:$M_1 \longrightarrow M_2$ be a bijection, then the following are equivalent

i. f is a homeomorphism
ii. f is a continuous open map
iii. f is a continuous closed open

Note 4 : Let f:$M_1 \longrightarrow M_2$ be a homeomorphism, f is a homeomorphism iff it satisfies the condition F is closed in M_1 iff f(F) is closed in M_1

Note 5 : Let f:$M_1 \to M_2$ is a bijection then f is a homeomorphism iffit satisfies the condition F is closed in M_1iff f(F) is closed in M_2

Examples: 1

The metric space [0,1] and [0,2] with usual metric are homeomorphic.

Proof:

Define f: [0,1] → [0,2] defined by f(x)=2x.

$f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$

∴ f is one-one

For all y∈ [0,2] there exist x∈ [0,1] such that $f(x) = y$

$\Rightarrow 2x = y$

$\Rightarrow x = \frac{y}{2} \in [0,1]$. ∴ f is onto

∴ f is bijection.

Clearly f is continuous.

Also f^{-1} (x) =$\frac{x}{2}$ is also continuous.

∴ f is homomorphism.

Examples:2

The metric space is (0,∞) and R with usual metrics are homeomorphic.

Proof: Define f: $(0,\infty)\rightarrow$R defined by f(x)=log_e x

Now , f(x)=f(y) $\Rightarrow log_e$ x=log_e y

$\Rightarrow e^{log_e x}=e^{log_e y}$

$\Rightarrow$ x=y

$\therefore$ f is one- one

For all y∈R there exist x∈(0,∞) such that f(x)=y

$\Rightarrow log_e$ x=y

$\Rightarrow e^{log_e x} = e^y$

$\Rightarrow$ x = e^y ∈(0,∞)

∴f is onto

∴ f is bijection

Clearly f: (0,∞)→R defined by f(x)=log_e x is continuous.

f^{-1}: (0,∞)→R defined by f(x)=e^x is continuous.

∴ f is homeomorphism.

Example :3

The metric space $({}^{-\pi}/_{2},{}^{\pi}/_{2})$ and R with usual metric are homeomorphic.

Proof: Define f: $({}^{-\pi}/_{2},{}^{\pi}/_{2})$ →R defined by f(x)=tanx.

f(x)=f(y) ⇒tanx =tany

⇒ x=y

∴ f is one- one

For all y∈R there exist x∈$({}^{-\pi}/_{2},{}^{\pi}/_{2})$ such that f(x)=y

⇒tanx=y ⇒ x = $tan^{-1}y$ ∈$({}^{-\pi}/_{2},{}^{\pi}/_{2})$

∴ f is onto

∴ f is bijection

Clearly f: $(^{-\pi}/_{2}, ^{\pi}/_{2})$→R defined by f(x)=tan x is continuous.

f^{-1}: R →$(^{-\pi}/_{2}, ^{\pi}/_{2})$ defined by f(x)=$tan^{-1}y$ is also continuous.

∴ f is homeomorphism.

Example:4

R with usual metric is not homeomorphic to R with discrete metric.

Proof: Let M_1=R with usual metric

Let M_2=R with discrete metric

Let f:$M_1 \longrightarrow M_2$ be any bijection

Now, {a}is open in M_2, but $f^{-1}(\{a\})$ is not open in M_1.

∴ f is not continuous

Thus any bijection f:$M_1 \longrightarrow M_2$ is not homeomorphism.

Hence M_1 is not homeomorphism to M_2

Example:5

The metric spaces (0,1) and (0,∞) with usual metric are homeomorphic.

Proof: Define f:(0,1) → (0,∞) defined by f(x) =$\frac{x}{1-x}$

f(x)=f(y) ⇒$\frac{x}{1-x} = \frac{y}{1-y}$

⇒ x(1-y) =y(1-x)

⇒x-xy = y-xy

⇒x=y

For all y∈(0,∞) there exist x∈(0,1) such that f(x)=y

$\Rightarrow\frac{x}{1-x}$ =y

$\Rightarrow\frac{x}{1-x}$ $=\frac{1}{y}$

$\Rightarrow\frac{1}{x}$ -1= $\frac{1}{y}$

$\Rightarrow\frac{1}{x}$ = $\frac{1}{y}$+1

$\Rightarrow$x=$\frac{y}{1+y}$

$\therefore$f is onto

$\therefore$ f is bijection

Clearly f is continuous.

f^{-1}: $\frac{x}{1-x}$ is also continuous.

$\therefore$ f is homeomorphism.

Definition: Let (M_1,d_1) and (M_2, d_2) be two metric spaces.Let f: $M_1\longrightarrow M_2$ be any bijection f is said to be isometry if $d_1(x,y) = d_2(f(x), f(y))$ for all x,y$\in M_1$

In other words an isometry is an distance preserving map M_1 and $M_2$are said to be isometric if there exists an isometry f:M_1onto M_2

Examples:

1. R^2 with usual metric and C with usual metric are isometric and f:$R^2\longrightarrow C$ defined by f(x,y)= x+iy is a required isometric.
2. Let d_1 be the usual metric on [0,1] and d_2 be the usual metric on [0,2] be map f:[0,1]⟶[0,2] defined by f(x)=2x is not a isometry.

$$\begin{aligned} d_2(f(x),f(y)) &= |f(x) - f(y)| \\ &= |2x - 2y| \\ &= 2|x - y| \\ &= 2d_1(x,y) \end{aligned}$$

$\therefore$ $d_1(x,y) \neq d_2(f(x),f(y))$

$\therefore$ f is not a isometry.

Note:

Since an isometry f preserves distances, the image of an open ball B(x,r) is the open ball B(f(x,r)).

Uniform Continuity:

Let (M_1,d_1) and (M_2,d_2) be two metric spaces. A function f:$M_1 \longrightarrow M_2$ is said to be uniformly continuous on M_1 if given $\varepsilon > 0$, there exists $\delta > 0$ such that $d_1(x,y)<\delta \Rightarrow d_2(f(x),f(y))< \varepsilon$.

Note:1

Uniform continuity is a global condition on the behaviour of a mapping on a set. Continuity is a local condition on the behaviour of function at a point.

Note 2: If f : $M_1 \rightarrow M_2$ is uniformly continuous on M_1, then f is continuous at every point of M_1, but a continuous function need not be uniformly continuous on M_1.

SOLVED PROBLEMS

Problem 1: Prove that f : [0,1] → R defined by f(x) = x^2 is uniformly continuous on [0,1].

Solution:

Let ε > 0 be given.

Let x, y ∈ [0,1]

Then x ≤ 1, y ≤ 1

$$|f(x) - f(y)| = |x^2 - y^2|$$
$$= |x - y|\,|x + y|$$
$$\leq 2\,|x - y|$$

Let δ = ε / 2

If $|x - y| < \delta=\frac{\varepsilon}{2} \Rightarrow |f(x) - f(y)| < \varepsilon$

∴ f is uniformly continuous on [0,1]

Problem 2 : Prove that f : (0,1)→R defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Solution:

Let ε > 0 be given.

Suppose there exists δ > 0 such that |x - y| < δ ⇒ |f(x) - f(y)| < ε

Given f(x) = 1/x,

$$\left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon$$

Take $x = y + \frac{\delta}{2} \Rightarrow x - y = \frac{\delta}{2}$

Clearly $|x - y| = \frac{1}{2}\delta < \delta$.

$\therefore |f(x) - f(y)| < \varepsilon$

$\therefore \left|\frac{1}{x} - \frac{1}{y}\right| < \varepsilon.$

$\therefore \left|\frac{1}{y + \frac{\delta}{2}} - \frac{1}{y}\right| < \varepsilon$

$$\therefore \left|\frac{y - \left(y + \frac{\delta}{2}\right)}{\left(y + \frac{\delta}{2}\right) y}\right| < \varepsilon$$

$$\therefore \left|\frac{\delta}{2\left(y + \frac{\delta}{2}\right) y}\right| < \varepsilon.$$

$$\therefore \left|\frac{\delta}{(2y + \delta)y}\right| < \varepsilon.$$

This inequality cannot be true for all y ∈ (0,1). Since $\frac{\delta}{(2y+\delta)y}$ becomes arbitrarily large as y approaches zero.

∴ f is not uniformly continuous.

Problem 3 : Prove that the function f :R → R defined by f(x) = sin x is uniformly continuous on R.

Solution:

Let x, y ∈R and x > y.

By Mean Value Theorem,

sin x - sin y = (x - y) cos z, where $x > z > y$.

⇒ |f(x) - f(y)| = |sin x - sin y|

= |x - y||cos z|

≤ |x - y| × 1

$= |x - y|$

Hence for a given $\varepsilon > 0$, if we choose $\delta = \varepsilon$, we have

$$|x - y| \leq \delta \Rightarrow |f(x) - f(y)| = |\sin x - \sin y| < \varepsilon$$

$\therefore$ $f(x) = \sin x$ is uniformly continuous on R.

Discontinuous functions on R

A function $f : R \to R$ is said to approach a limit l as $x \to a$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x\to a} f(x) = l$.

Definition:

A function f is said to have l as the right limit at $x = a$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $a < x < a + \delta \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x\to a+} f(x) = l$.

Also, we denote the right limit by $f(a+)$.

A function f is said to have l as the left limit at $x = a$ if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $a - \delta < x < a \Rightarrow |f(x) - l| < \varepsilon$ and we write $\lim_{x\to a-} f(x) = l$.

Also, we denote the left limit by $f(a-)$.

Notes:

1. $\lim_{x\to a} f(x)$ exists if and only if the left and right limits of $f(x)$ at $x = a$ exist and are equal.

2.The definition of continuity of f at $x = a$ can be formulated as follows:

f is continuous at a if and only if $f(a+) = f(a-) = f(a)$.

3. If $\lim_{x\to a} f(x)$ does not exist, then one of the following must hold:

(i) $\lim_{x\to a^+} f(x)$ does not exist.

(ii) $\lim_{x\to a-} f(x)$ does not exist.

(iii) $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a-} f(x)$ exist but are unequal.

Definition If a function f is discontinuous at a, then a is called a point of discontinuity for the function.

If a is a point of discontinuity of a function f, then any one of the following cases arises:

(i) $\lim_{x\to a} f(x)$ exists but is not equal to $f(a)$.

(ii) $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a-} f(x)$ exist and are not equal.

(iii) Either $\lim_{x\to a-} f(x)$ or $\lim_{x\to a^+} f(x)$ does not exist.

Definition : Let a be a point of discontinuity for f(x), then a is said to be a point of discontinuity of the first kind if $\lim_{x\to a+} f(x)$ and $\lim_{x\to a-} f(x)$ exist and both of them are finite and unequal.

The point a is said to be a point of discontinuity of the second kind if either if $\lim_{x\to a+} f(x)$or if $\lim_{x\to a-} f(x)$ does not exist or is infinite or if $\lim_{x\to a-} f(x)$ does not exists.

Definition : Let A ⊆R. A function f: A → R is called "monotonic increasing" if $x < y \Rightarrow f(x) \leq f(y)$.

f is called monotonic decreasing if $x > y \Rightarrow f(x) \geq f(y)$.

A function f is called "monotonic" if it is either monotonic increasing or monotonic decreasing.

Theorem 3.5 Let f: [a, b] → R be a monotonic increasing function.Then f has a left limit at every point of (a, b). Also f has a right limit at a and f has a left limit at b. Further x < y ⇒f(x +) ≤ f(y –). Similar result is true for monotonic decreasing function.

Proof: Let f: [a, b] → R be monotonic increasing.

Let x ∈ [a, b].

Then {f(t): a ≤ t < x} is bounded above by f(x).

Let l = l.u.b {f(t): a ≤ t < x}

We claim that f(x–) = l.

Let ε > 0 be given. By definition of l.u.b there exists t such that a ≤ t < x and

l – ε < f(t) ≤ l.

∴t< u < x ⇒ l – ε < f(t) ≤ f(u) ≤ f(x)

(∵ f is monotonic increasing).

⇒ l – ε < f(u) ≤ l

∴ x – δ < u < x ⇒ l – ε < f(u) ≤ l where δ = x – t

$\therefore f(x-) = l$.

Similarly, we can prove that $f(x+) = \text{g.l.b } \{f(t): x \le t \le b\}$

Similarly, let f: [a, b] → R be monotonic decreasing.

Let $x \in [a, b]$.

Then $\{f(t): x < t \le b\}$ is bounded below by f(x).

Let $l = \text{g.l.b } \{f(t): x < t \le b\}$.

We claim that $f(x+) = l$.

Let $\varepsilon > 0$ be given. By definition of g.l.b there exists t such that

$x < t \le b$ and $l \le f(t) < l + \varepsilon$.

$\therefore t < u < x \Rightarrow l \le f(t) < f(u) < l + \varepsilon$ ($\because$ f is monotonic decreasing)

$\Rightarrow l \le f(u) < l + \varepsilon$

$\therefore x < u < x + \delta \Rightarrow l \le f(u) < l + \varepsilon$ where $\delta = x - t$.

$\therefore f(x+) = l$.

Now we shall prove that $x < y \Rightarrow f(x+) \le f(y-)$

Let $x < y$

$f(x+) = \text{g.l.b } \{f(t): x \le t \le y\}$

$= \text{g.l.b } \{f(t): x < t \le y\}$ ………………(1)

$f(y-) = \text{l.u.b } \{f(t): a \le t < y\}$

$= \text{l.u.b } \{f(t): x \le t < y\}$ ………………(2)

From (1) & (2) we get $f(x+) \le f(y-)$

Theorem 3.6 : Let f: [a, b] → R be a monotonic function. Then the set of points of [a, b] at which f is discontinuous is countable.

Proof: Let $E = \{x: x \in [a, b]$ and f is discontinuous at $x\}$

Let x ∈ E

f(x+) and f(x−) exists andf(x−) ≤ f(x+)

If f(x−) = f(x+), then f(x−) = f(x+) = f(x)

∴ f is continuous at x, which is a contradiction to f(x−) ≠ f(x+)

∴ f(x−) < f(x+)

Now choose a rational number r(x) such that f(x−) < r(x) < f(x+)

This defines a map r: E → Q which maps x to r(x)

We claim that r is 1–1.

Let $x_1 < x_2$.

By the previous theorem: x < y ⇒ f(x+) ≤ f(y−)

∴ x_1 <x_2. ⇒ f(x_1+) ≤ f(x_2−)

Also $f(x_1 -) < r(x_1) < f(x_1 +) and\, f(x_2 -) < r(x_2) < f(x_2 +)$.

$$\therefore r(x_1) < f(x_1 +) < f(x_2 -) < r(x_2)$$

Thus $x_1 < x_2 \Rightarrow r(x_1) < r(x_2)$.

∴ $r: E \rightarrow Q$ is 1–1

Hence E is countable.

Definition : A subset D of R is said to be of type Fσ if D can be expressed as the countable union of closed sets.i.e., D =$U_{n=1}^{\infty}\ F_n$ where F_n is a closed subset of R.

Notes:

1. Any closed subset F is of type Fσ since F=$U_{n=1}^{\infty} F_n$ where F_n = F for all n.

2. A set of type Fσ need not be closed.

Example: Q is of type F_σ but Q is not closed.

Definition : Consider any function f: R → R. Let I be a bounded open interval in R. Then the oscillation of f over I is denoted by ω(f, I) and it is defined by

$$\omega(f, I) = l.u.b\,\{f(x): x \in I\} - g.l.b\,\{f(x): x \in I\}.$$

If a ∈R, then the oscillation off at a is denoted by ω(f, a)and it is defined by $\omega(f, a) = g.l.b\omega(f, I)$ where the g.l.b is taken over all bounded open intervals containing a.

Note:

1. For any n ∈ Z, $\omega(f, n) = 1$.

2. From the definition $\omega(f, I) \geq 0$ for any I.

   Hence $\omega(f, a) \geq 0$ for any a ∈ R.

Theorem 3.7 : A function f: R → R is continuous at a ∈ R iff $\omega(f, a) = 0$.

Proof:

Suppose f is continuous at a.

To prove $\omega(f, a) = 0$.

Let ε > 0 be given.

Then there exists δ > 0 such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon/2$

$$Let\ I = (a - \delta, a + \delta)$$

For any $x \in I, |f(x) - f(a)| < \frac{\varepsilon}{2}$

$$|f(x) - f(y)| = |f(x) - f(a) + f(a) - f(y)|$$

$$\leq |f(x) - f(a)| + |f(y) - f(a)|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

For any $x, y \in I, |f(x) - f(y)| < \varepsilon$

$$\therefore \omega(f, I) < \varepsilon$$

$$\therefore g.l.b\omega(f, I) < \varepsilon$$

$$i.e., \omega(f,a) < \varepsilon$$

Since ε > 0 is arbitrary, $\omega(f,a) = 0$.

Conversely, assume that $\omega(f,a) = 0$.

To prove f is continuous at a.

Let ε > 0 be given.

We have $\omega(f,a) = 0$.

$$\Rightarrow g.l.b\omega(f,I) = 0.$$

∴ There exists a bounded open interval I containing a such that $0 < \omega(f,I) < \varepsilon$.

Let $x_1, x_2 \in$ I.

Then f$(x_1) \leq l.u.b\{ f(x) : x \in I \} and\, f(x_2) \geq g.l.b\{ f(x) : x \in I \}$

$$\Rightarrow -f(x_2) \leq -g.l.b\{ f(x) : x \in I \}$$

$$\Rightarrow |f(x_1) - f(x_2)| \leq l.u.b\{ f(x) : x \in I \} - g.l.b\{ f(x) : x \in I \}$$

$$= \omega(f,I) < \varepsilon$$

$$\therefore |f(x_1) - f(x_2)| < \varepsilon$$

In particular, $|f(x) - f(a)| < \varepsilon\, for\, all\, x \in I$.

Since I is a bounded open interval containing a, we can choose δ > 0 such that

$$(a - \delta, a + \delta) \subseteq I.$$

$$\therefore |f(x) - f(a)| < \varepsilon\, for\, all \in (a - \delta, a + \delta)$$

$$\therefore |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$$

∴ f is continuous at a.

Theorem 3.8 : Let f : R → R be any function. $Let\, r > 0$.

$$Then\, E_r = \{a \in R \mid \omega(f,a) \geq 1/r\}\, is\, a\, closed\, set.$$

Proof: Let x be any limit point of E_r.

We claim that x ∈ E_r.

For this, we must prove that $\omega(f, x) \geq 1/r$.

Now let I be any bounded open interval containing x.

Since x is the limit point of E_r, I contains a point y of E_r.

Hence I is a bounded open interval containing y.

$$\therefore \omega(f, y) \leq \omega(f, I)$$

$$Since\ y \in E_r, \omega(f, y) \geq 1/r$$

$$\therefore\ we\ have\ \omega(f, I) \geq \omega(f, y) \geq 1/r$$

This is true for any bounded open interval I containing x.

$$\therefore \omega(f, x) \geq 1/r$$

$$\therefore x \in E_r$$

∴ E_r contains all its limit points.

Hence E_r is closed.

Theorem 3.9 : Let D be the set of points of discontinuities of a function f : R → R. Then D is of type Fσ.

Proof: Let D be the set of points of discontinuities of f.

To prove: D is of type Fσ.

i.e., To prove D = $U_{n=1}^{\infty} E_n$, where E_n = {a ∈ R | $\omega(f, a) \geq 1/n$} is closed.

Let x ∈ D.

∴ f is discontinuous at x.

∴$\omega(f, x) > 0$.

⇒$\omega(f, x) \geq 1/n$ for some n > 0.

∴ x ∈E_n for some n > 0.

$\therefore$ x $\in \bigcup_{n=1}^{\infty} E_n$.

x ∈ D ⇒ x $\in \bigcup_{n=1}^{\infty} E_n$

$\therefore$ D $\subseteq \bigcup_{n=1}^{\infty} E_n$ ……………… (1)

Let x $\in \bigcup_{n=1}^{\infty} E_n$

x ∈ E_n for some positive integer n.

$\therefore \omega(f, x) \geq 1/n$ for some n > 0

Hence ω(f, x) > 0

∴ f is discontinuous at x.

Hence x ∈ D

x $\in \bigcup_{n=1}^{\infty} E_n$ ⇒ x ∈ D

$\therefore \bigcup_{n=1}^{\infty} E_n \subseteq$ D — (2)

From (1) & (2),

D = $\bigcup_{n=1}^{\infty} E_n$

Also each E_n is closed.

Thus D is a countable union of closed sets.

∴ D is of type Fσ

Theorem 3.10 : There is no function f : R → R such that f is continuous at each rational number and discontinuous at irrational number.

Proof: Suppose A is of type Fσ.

Then A = $\bigcup_{n=1}^{\infty} F_n$ where each Fn is closed.

Now, since Fn contains only irrational number, Fn cannot contain any open interval.

∴ Int Fn = φ

Int $\overline{Fn}$ = φ [∵Fn is closed]

$\therefore F_n$ is nowhere dense

$\therefore$ A is of first category, which is a contradiction.

$\therefore$ A is not of type Fσ.

Hence, the theorem.

Exercises

1. Let $f: R \to R$ be defined by $f(x) = \begin{cases} -2 \ if \ x < 0 \\ 2 \ if \ x \geq 0 \end{cases}$. Prove that f is not continuous.
2. Let (M,d) be any metric space. Let $f: M \to R$ and g: $M \to R$ be any two continuous functions. Define
   (i) $(fg)(x) = f(x)g(x)$
   (ii) $(cf)(x) = cf(x) \ where \ c \in R.$
   (iii) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ if g(x)≠0 for all x∈ M.

   Prove that fg. cf and f/g are all continuous.

3. Give an example of a map from R to itself which is continuous and closed but not an open map. [Hint : Consider any constant map]
4. Let (M,d) be any metric space. Prove that the identity map $i: M \to M$ is a homeomorphism.
5. Determine which of the following functions are uniformly continuous.
   (i) f: $R \to R$ defined by $f(x) = kx \ where \ k \in R.$
   (ii) f: $R \to R$ defined by $f(x) = x^3$.
   (iii) f: $(0,1) \to R$ defined by $f(x) = \frac{1}{1-x}$.
6. Let $f: R \to R$ and $g: R \to R$ be two functions uniformly continuous on R. Prove that $f + g$ is also uniformly continuous on R.
7. Is the product of uniformly continuous real valued functions again uniformly continuous?

UNIT IV

CONNECTEDNESS

Definition: Let (M, d) be a metric space, then M is said to be connected if M cannot be expressed as the union of two disjoint non-empty open sets. If M is not connected it is said to be disconnected.

Examples:

1. Let M= [1,2] U [3,4] with usual metric. Then M is disconnected.

   Proof: [1,2] & [3,4] are open in M.

   Also, A=[1,2] $\neq \Phi$; B =[3,4] $\neq \Phi$ & $A \cap B = \Phi$

   Thus, M is the Union of two disjoint non-empty open sets.

2. In a discreate metric space M with more than one point is disconnected.

   Proof: Let A be a proper non-empty subset of M.

   Since M has more than one point such a set exist.

   The A^C is also non-empty.

   Since M is discrete every subset of M is open.

   $\therefore$A& A^C are open.

   Thus, M=AU A^C, where A& A^C are two disjoint non-empty open set.

   $\therefore$M is not connected.

THEOREM 4.1: Let (M, d) be a metric space. Then the following are equivalent.

(i) M is connected.

(ii) M cannot be written as the union of two disjoint non-empty closed sets.

(iii) M cannot be written as the union of two non-empty sets A&B such that $A \cap \overline{B} = \overline{A} \cap B = \Phi$.

(iv) M& Φ are the only sets which are both open & closed in M.

Proof: (i)=>(ii)

Assume that M is connected.

To Prove: M cannot be written as the union od two disjoint non-empty closed sets.

Suppose M can be written as the union of two disjoint non-empty closed sets.

∴M=AUB, where A&B are closed,

$A\neq \Phi$, $B\neq \Phi$ & $A\cap B = \Phi$.

Since, $A\cap B=\Phi$, $A^c=B$, $B^c=A$.

Since, A&B are closed, A^C & B^C are open.

∴ B & A are open.

∴ M=AUB, where A&B are open, $A\neq \Phi$, $B\neq \Phi$ & $A\cap B = \Phi$.

∴ M is disconnected. which is a contradiction.

∴ Our assumption is wrong.

Hence, M cannot be written as the union of two disjoint non-empty closed sets.

(ii)=>(iii)

Assume that M cannot be written as the union of two disjoint non-empty closed sets.

To Prove: M cannot be written as the union of two disjoint non-empty sets A&B such that $A\cap\overline{B}= \overline{A}\cap B = \Phi$.

Suppose M can be written as the union of two disjoint non-empty sets A&B such that $A\cap\overline{B}= \overline{A}\cap B = \Phi$.

We claim that A & B are closed sets.

i.e.)., To Prove: $A=\overline{A}$ & $B=\overline{B}$.

Let $x\in\overline{A}$

We have $\overline{A}\cap B= \Phi$.

$\therefore x \notin B$

$\therefore x \in A$ (since, AUB=M)

$\therefore \overline{A}\subseteq A$

Always $A \subseteq \bar{A}$

Hence, $A = \bar{A}$

Let $x \in \bar{B}$

We have $A \cap \bar{B} = \Phi$.

$\therefore x \notin A$

Since $A \cup B = M$, $x \in B$, $x \notin A$

$\therefore \bar{B} \subseteq B$

Always $B \subseteq \bar{B}$

$\therefore B = \bar{B}$

Also, $A \cap \bar{B} = \bar{A} \cap B = \Phi$.

$\therefore$ M can be written as the union of two disjoint non-empty sets, which is acontradiction to our assumption.

Hence, M cannot be written as the union of two non-empty sets A&B such that $A \cap \bar{B} = \bar{A} \cap B = \Phi$.

(iii)=>(iv)

Assume that M cannot be written as the union of two non-empty sets A&B such that $A \cap \bar{B} = \bar{A} \cap B = \Phi$.

To prove M & Φ are the only sets which are both open & closed in M.

Suppose M & Φ are the only sets which are both open & closed in M is not true.

Then there exists $A \subseteq M$ such that $A \neq M$ & $A \neq \Phi$ & A is both open and closed.

Let $B = A^C$

Then B is also both open and closed & $B \neq \Phi$.

Also, $M = A \cup B$

Further, $\bar{A} \cap B = A \cap A^C = \Phi$. [since, A is closed, $A = \bar{A}$ & $B = A^C$]

Similarly,

If $A = B^C$, then $A \cap \bar{B} = \Phi$

$\therefore$ $M = A \cup B$, where $A \cap \bar{B} = \bar{A} \cap B = \Phi$.

which is contradiction to (iii)

$\therefore$ Our assumption is wrong.

Hence (iii)=>(iv).

(iv)=>(i)

Assume that M is connected.

Prove that M & Φ are the only sets which are both open & closed in M.

Suppose M is not connected.

$\therefore$ $M = A \cup B$, where $A \neq \Phi$, $B \neq \Phi$, A& B are open and $A \cap B = \Phi$.

Then $B^C = A$.

Now, Since B is open, B^C is closed.

$\therefore$ A is closed.

Also, $A \neq \Phi$ & $A \neq M$.

$\therefore$ A is the proper non-empty subset of M which is both open & closed which is a contradiction to both.

Hence, (iv)=>(i).

EQUIVALENT CHARACTERIZATIONS FOR COMPACTNESS

Theorem 4.2 : A metric space M is connected iff there does not exist a continuous function f: M onto the discrete metric space {0, 1}.

Proof: Suppose M is connected.

To prove: there does not exist a continuous function f: M onto the discrete metric space {0, 1}.

Suppose there exists a continuous function f: M onto {0, 1}.

Since {0, 1} is discrete, {0} and {1} are open.

∴A $= f^{-1}$ ({0}) and B = f^{-1}({1}).

We know that: Inverse image of every open set is open.

∴ f^{-1} ({0}) and f^{-1} ({1}) are open in A.

∴ A and B are open in M.

Since f is onto, A and B are non-empty.

Clearly, A ∩ B = ∅ and A ∪ B = M.

Thus, M = A ∪ B, where A and B are disjoint non-empty open sets.

∴ M is not connected, which is a contradiction.

Hence, there does not exist a continuous function {0, 1}.

Conversely, Assume that there does not exist a continuous function f: M onto the discrete metric space {0, 1}.

To Prove: M is connected.

Suppose M is not connected.

Then there exist disjoint non-empty open sets A and B in M such that M = A ∪ B.

Now, define f: M → {0, 1} at $f(x) = \begin{cases} 0 & if\ x \in A \\ 1 & if\ x \in B \end{cases}$

Clearly, f is onto.

Also, f^{-1} (∅) = ∅, f^{-1} ({0}) = A, and f^{-1} ({1}) = B, f^{-1} ({0, 1}) = M.

Thus, the inverse image of every open set in {0, 1} is open in M.

Hence, f is continuous, which is a contradiction to our assumption.

Thus, M is connected.

Note : The above theorem can be restated as follows:

M is connected iff every continuous function f: M → {0, 1} is not onto.

SOLVED PROBLEMS

Problem 1 : Let M be a metric space. Let A be a connected subset of M. If B is the subset of M such that $A \subseteq B \subseteq \bar{A}$, then B is connected. In particular, $\bar{A}$ is connected.

Solution: Let M be a metric space. Let A be a connected subset of M. If B is the subset of M such that $A \subseteq B \subseteq \bar{A}$.

To prove B is connected.

Suppose B is not connected.

Then $B = B_1 \cup B_2 \, where \, B_1 \neq \emptyset, B_2 \neq \emptyset, B_1 \cap B_2 = \emptyset$ and B_1, B_2 are open in B.

Now, since $B_1 \, and \, B_2$ are open sets in B, there exist open sets $G_1 \, and \, G_2$ in M such that $B_1 = G_1 \cap B \, and \, B_2 = G_2 \cap B$.

$$\therefore B = B_1 \cup B_2 = (G_1 \cap B) \cup (G_2 \cap B) = (G_1 \cup G_2) \cap B.$$

$$\therefore B \subseteq G_1 \cup G_2.$$

$\therefore A \subseteq G_1 \cup G_2$ [Since $A \subseteq B$].

$$\therefore A = (G_1 \cup G_2) \cap A = (G_1 \cap A) \cup (G_2 \cap A).$$

Now, $G_1 \cap A \, and \, G_2 \cap A$ are open in A.

Further, $(G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A.$

$$= (G_1 \cap G_2) \cap A \quad [Since \, A \subseteq B]$$

$$= (G_1 \cap B) \cap (G_2 \cap B)$$

$$= B_1 \cap B_2$$

$$= \emptyset.$$

$$\therefore (G_1 \cap A) \cap (G_2 \cap A) = \emptyset.$$

Now, since A is connected, either $G_1 \cap A = \emptyset$ or $G_2 \cap A = \emptyset$.

Without loss of generality, let us assume that $G_1 \cap A = \emptyset$.

Since G_1 is open in M, we have $G_1 \cap \bar{A} = \emptyset$.

$$\therefore G_1 \cap B = \emptyset. \qquad [since, B \subseteq \bar{A}]$$

$\therefore B_1 = \emptyset$, which is a contradiction.

∴ B is connected.

In particular, $\bar{A}$ is also connected.

Problem 2 : If A and B are connected subsets of a metric space M and if A ∩B≠Φ, prove that A∪B is connected.

Solution: Let f: A∪B→{0,1} be a continuous function.

Since, A ∩B≠Φ, we can choose x_0∈ A ∩ B.
Let $f(x_0)=0$.
∴ f: A ∪ B → {0,1} is continuous f/A: A →{0,1}is also continuous.
But A is connected.
Hence f/A is not onto.
∴ f(x) = 0 for all x ∈ A or f(x) = 1 for all x ∈ A.
But $f(x_0) = 0$ and x_0 ∈ A.
∴f(x) = 0 for all x ∈ A.
Similarly,
f(x) = 0 for all x ∈B.
∴f(x) = 0 for all x ∈ AUB.
Thus, any continuous function f: A ∪ B → {0,1}is not onto.
∴ A ∪ B is connected.

CONNECTED SUBSETS OF R

Theorem 4.3 : A subspace of R is connected iff it is an interval.

Proof: Let A be a connected subset of ℝ.

Suppose A is not an interval.

Then there exists $a, b, c \in \mathbb{R}$ such that a < b < c and a, c ∈ A, but b ∉ A.

Let $A_1 = (-\infty, b) \cap A$ and $A_2 = (b, \infty) \cap A$.

Since (-∞, b) and (b, ∞) are open in $\mathbb{R}, A_1$ and A_2 are open sets in A.

Also, $A_1 \cap A_2 = \emptyset$ and $c \in A_2$.

Further, $a \in A_1$ and $c \in A_2$.

Hence $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$.

Thus, A is the union of two disjoint non-empty open sets A_1 and A_2.

Hence A is not connected, which is a contradiction.

Hence A is an interval.

Conversely, Let A be an interval.

We claim that A is connected.

Suppose A is not connected.

Let $A = A_1 \cup A_2$ $where$ $A_1 \neq \emptyset, A_2 \neq \emptyset, A_1 \cap A_2 = \emptyset$ and A_1, A_2 are closed sets in A.

Choose $x \in A_1$ and $z \in A_2$.

Since $A_1 \cap A_2 = \emptyset$, we $have$ $x \neq z$.

Either $x < z$ or $z > x$. Without loss of generality, we assume that $x < z$.

Now, since A is an interval, we have [x, z] ⊆ A.

i.e., $[x, z] \subseteq A_1 \cup A_2$.

∴Every element of [x, z] is either in A_1 or in A_2.

Now let $y = l.u.b\{[x, z] \cap A_1\}$.

Clearly $x \leq y \leq z$.

Hence y ∈ A.

Let ε > 0 be given. Then by the definition of l.u.b, there exists $t \in [x,z] \cap A_1$ $such\ that\ y - \varepsilon < t \leq y$.

$$\therefore (y - \varepsilon, y + \varepsilon) \cap ([x,z] \cap A_1) \neq \emptyset$$

$$\Rightarrow y \in \overline{[x,z] \cap A_1}$$

$$y \in [x,z] \cap A_1 \quad [since, [x,z] \cap A_1\ is\ closed\ in\ A]$$

$\therefore y \in A_1$ ……………….. (1)

Again, by the definition of $y, y + \varepsilon \in A_2\ for\ all\ \varepsilon > 0\ such\ that\ y + \varepsilon \leq z$

$$\therefore y \in \bar{A_2}$$

$$\therefore y \in A_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots. (2)(since\ A_2\ is\ closed)$$

$$y \in A_1\ and\ y \in A_2$$

$$y \in A_1 \cap A_2\ [by\ (1)\ \&\ (2)]$$

which is a contradiction.

Since $A_1 \cap A_2 \neq \emptyset$,

Hence A is connected.

Theorem 4.4 : R is connected.

Proof: By previous theorem, a⊆ R is connected if it is an interval.

We have R = (-∞, ∞) is an interval.

∴ R is connected.

SOLVED PROBLEMS

Problem 1: Give an example to show that a subspace of a connected metric space need not be connected.

Solution: We know that R is connected.

Let A = [1,2] ∪ [3,4] is a subspace of R.

But A = [1,2] ∪ [3,4] is not connected.

∴A subspace of a connected metric space need not be connected.

Problem 2 : Prove or disprove: If A and C are connected subsets of a metric space M and if $A \subseteq B \subseteq C$, then B is connected.

Solution:

We disprove the statement by giving a counterexample.

Let A = [1,2], B = [1,2] ∪ [3,4], C = R.

Clearly, $A \subseteq B \subseteq C$

Here A and C are connected, but B is not connected.

CONNECTEDNESS AND CONTINUNITY

Theorem 4.5 : Let M_1 be a connected metric space. Let M_2 be any metric space.

Let f: $M_1 \rightarrow M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2.

(i.e.) Any continuous image of a connected set is connected.

Proof:

$$Let\ f: M_1 \rightarrow M_2\ be\ a\ continuous\ function\ and\ M_1\ be\ a\ connected\ metric\ space.$$

Let $f(M_1) = A$.

So that f is a function from M_1 onto A.

We claim that A is connected.

Suppose A is not connected.

Then there exists a proper non-empty subset B(A) which is both open and closed in A.

∴ $f^{-1}(B)$ is a proper non-empty subset of M_1 which is both open and closed in M_1.

Hence, M_1 is not connected, which is a contradiction. [since, M_1 is connected].

∴ A is connected.

Theorem 4.6 : INTERMEDIATE VALUE THEOREM

Statement : Let f be a real-value continuous function defined on an interval I. Then f takes every value between any two values it assumes.

Proof: Let f be a real-value continuous function defined on an interval I.

Let a, b ∈ I and let $f(a) \neq f(b)$.

Then either $f(a) < f(b)$ or $f(a) > f(b)$.

Without loss of generality, we assume that $f(a) < f(b)$.

Let c be such that $f(a) < c < f(b)$.

The interval I is a connected subset of R.

Since f is continuous and I is a connected subset of R $f(I)$ is a connected subset of R. [Since, the continuous image of a connected set is connected].

We have a subspace of R is connected iff it is an interval.

$\therefore f(I)$ is an interval.

Also, $f(a), f(b) \in f(I)$.

Hence, $[f(a), f(b)] \subseteq f(I)$.

$$\therefore c \in f(I)$$

$$\therefore c = f(x) \text{ for some } x \in I.$$

Thus, f takes every value between any two values.

SOLVED PROBLEMS

Problem 1 : Prove that if f is a non-constant real-valued continuous function on R, then the range of f is uncountable.

Solution: We know that R is connected.

Since f is a continuous function on R, f(R) is a connected subset of R. (Since, Continuous image of a connected set is connected.)

∴ f(R) is an interval in R.

Also, since f is a non-constant function, the interval f(R) contains more than one point.

∴ f(R) is uncountable.

Thus, the range of f is uncountable.

NOTES :

1. Q (the set of rational numbers) is not connected.
2. If M is a metric space and x ∈ M, then {x} is a connected subset of M.
3. A subset of a discrete metric space is connected iff it is a {}.

Exercises :

1. Prove that [0,1] is not a connected subset of R with discrete metric.
2. Prove that any connected subset of R containing more than one point is uncountable. [Hint : Any interval containing more than one point is uncountable]
3. Determine which of the following statements are true and which are false.
   (i) R is connected
   (ii) Q is connected.
   (iii) A subspace of a connected space is connected.
   (iv) If A and B are connected subsets of a metric space M then AUB is connected.
   (v) Any discrete metric space having more than one point is disconnected.

CONTRACTION MAPPING THEOREM

Definition: Let (M, d) be a metric space. A mapping T: M → M is called a contraction mapping if there exists a positive real number α < 1 such that

$$d(T(x), T(y)) \leq \alpha\, d(x, y) \; for\ all\ x, y \in M.$$

Note: If T is a contraction mapping, then the distance d(T(x), T(y)) is less than the distance d(x, y).

Example 1: Let T: [0, 1/3] → [0, 1/3] defined by T(x) = x^2 is a contraction mapping.

Solution: Let $x, y \in [0, 1/3]$.

$$Then\ x \leq 1/3\ and\ y \leq 1/3.$$

$$\begin{aligned} d(T(x), T(y)) &= |T(x) - T(y)| \\ &= |x^2 - y^2| \\ &= |(x + y)(x - y)| \\ &\leq \left(\frac{2}{3}\right) |x - y|. \\ &= \left(\frac{2}{3}\right) d(x, y) \end{aligned}$$

Here, $\alpha = 2/3 < 1$.

$$\therefore\ d(T(x), T(y)) \leq \left(\frac{2}{3}\right) d(x, y)$$

∴ T is a contraction mapping.

Example 2 : T: R → R be defined by $T(x) = \frac{1}{2}x$ is a contraction mapping since $d\big(T(x),T(y)\big) = \frac{1}{2}d(x,y)$.

Example 3 : The function $T: l_2 \to l_2$ defined by $T(x) = (x_n / 2)\ is\ a\ contraction\ mapping, where\ x = (x_n)$.

Solution: $Let\ x, y \in l_2$.

Then $x = (x_n)\ and\ y = (y_n)$.

$$d\big(T(x),T(y)\big) = \left[\sum_{n=1}^{\infty}(T(x) - T(y))^2\right]^{1/2}$$

$$= \left[\sum_{n=1}^{\infty}((1/2)x_n - (1/2)y_n)^2\right]^{\frac{1}{2}}$$

$$= \left[\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^2(x_n - y_n)^2\right]^{1/2}$$

$$= \left(\tfrac{1}{2}\right)\left[\textstyle\sum_{n=1}^{\infty}(x_n - y_n)^2\right]^{1/2}$$

= (1/2) d(x, y)

∴ d(T(x), T(y)) = (1/2) d(x, y)

∴ T is a contraction mapping.

Example 4 : Let T: [0,1] → [0,1] be a differentiable function. If there is a real number αwith 0 < α < 1 such that |T′(x)| ≤ α for all x ∈ [0,1], where T′ is the derivative of T, then T is a contraction mapping.

Solution: Let x, y ∈ [0,1] with x < y.

By Mean Value Theorem,

$$T(y) - T(x) = (y - x)\,T'(x)$$

$$|T(y) - T(x)| = |y - x|\,|T'(x)|$$

$$\leq \alpha|y - x|$$

$$\therefore\ d(T(y),T(x)) \leq \alpha\, d(y,x)\ where\ 0 < \alpha < 1.$$

THEOREM 4.7 : Let T: M → M be a contraction mapping. Then T is uniformly continuous on M.

Proof: By the definition of a contraction mapping,

$$d(T(x),T(y)) \leq \alpha\, d(x,y)\ where\ \alpha < 1.$$
$$\therefore d(T(x),T(y)) < d(x,y)$$
$$Let\ \varepsilon > 0\ be\ given.$$

Choose $\delta = \varepsilon$.

$$d(x,y) < \delta \Rightarrow d(T(x),T(y)) < \varepsilon.$$

∴ T is uniformly continuous on M.

CONTRACTION MAPPING THEOREM

Statement : Let (M, d) be a complete metric space.Let T: M → M be a contraction mapping. Then there exists a unique point x in M such that T(x) = x.

(i.e.) T has exactly one fixed point.

Proof: Let x_0 be the arbitrary point in M. Let $x_1 = T(x_0)$, $x_2 = T(x_1)$, …, $x_n = T(x_{n-1})$.

We claim that, (x_n) is a Cauchy sequence in M.

Since, T is the contraction mapping, there exists a positive real number α, such that $0 < \alpha < 1\ and\ d(T(x),T(y)) < \alpha\, d(x,y)$.

$$\begin{aligned} \therefore\ d(x_n,x_{n+1}) &= d(T(x_{n-1}),T(x_n)) \\ &\leq \alpha\, d(x_{n-1},x_n) \\ &\leq \alpha^2 d(x_{n-2},x_{n-1}) \\ &\leq \alpha^3 d(x_{n-3},x_{n-2}) \\ &\ \ \vdots \\ &\leq \alpha^{n} d(x_0,x_1). \end{aligned}$$

$$\therefore\ d(x_n,x_{n+1}) \leq \alpha^{n} d(x_0,x_1) \qquad \dots\dots\dots\dots \qquad (1)$$

$$Now, let\ m,n \in \mathbb{N}\ and\ m > n.$$

$$Then, d\,(x_n,x_m) \leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{m-1},x_m)$$

$$\leq \alpha^{n} d(x_0,x_1) + \alpha^{n+1} d(x_0,x_1) + \cdots + \alpha^{m-1} d(x_0,x_1)\ \text{[using (1)}$$

$$= \alpha^{n} d(x_0, x_1) [1 + \alpha + \alpha^2 + \cdots + \alpha^{m-n-1}]$$

$$< \alpha^{n} d(x_0, x_1) \left[\frac{1}{(1 - \alpha)}\right]$$

$$\therefore d\ (x_n, x_m) < \left(\frac{\alpha^{n}}{1 - \alpha}\right) d(x_0, x_1) \quad \ldots\ldots\ldots\ldots\ldots \quad (2)$$

Since $0 < \alpha < 1$, *the sequence* $(\alpha^{n}) \to 0$.

∴ Given ε > 0, there exists a positive integer n_1 such that $|(\alpha^{n} / (1 - \alpha))\, d(x_0, x_1)| < \varepsilon$ *for all* $n \geq n_1$.

$$\therefore (2) \Rightarrow d\ (x_n, x_m) \leq \varepsilon \text{ for all } m, n \geq n_1.$$

Hence (x_n) is a Cauchy sequence in M.

Since M is complete, there exists x ∈ M such that $((x_n)) \to$ x.

Also, T is continuous.

Hence, $(T(x_n)) \to T(x)$.

$\therefore T(x) = \lim_{n\to\infty} T(x_n)$

$= \lim_{n\to\infty} x_{n+1}$

= x.

Thus, $T(x) = x$.

Hence, x is the fixed point of T.

Now, to prove the uniqueness:

Suppose, there exists y ∈ M such that y ≠ x and T(y) = y.

Then, $d(x, y) = d\big(T(x), T(y)\big)$

$\leq \alpha\, d(x, y)$

$$\Rightarrow d(x, y) - \alpha\, d(x, y) \leq 0$$

$$\Rightarrow d(x, y)(1 - \alpha) \leq 0$$

$$Clearly, d(x, y) > 0.$$

$$Also, \alpha < 1$$

$$\therefore 0 < 1 - \alpha.$$

$$i.e. (1 - \alpha) > 0.$$

$\therefore d(x, y)(1 - \alpha) > 0$, which is a contradiction.

$$\therefore y = x$$

Hence, x is the unique fixed point of T.

UNIT V

COMPACTNESS

Compact Metric Spaces

Definition : Let M be a metric a metric space. A family of open sets $\{G_\alpha\}$ in M is called an **open cover** for M if $\bigcup G_\alpha = M$.

A subfamily of $\{G_\alpha\}$ which itself is an open cover is called a **subcover.**

A metric space M is said to be **compact** if every open cover for M has a finite subcover.

i.e., for each family of open sets $\{G_\alpha\}$ such that $\bigcup G_\alpha = M$, there exists a finite subfamily $\{G_{\alpha 1}, G_{\alpha 2}, \ldots, G_{\alpha n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha i} = M$.

Example 1 : R with usual metric is not compact.

Proof: Consider the family of open intervals {(-n, n): n\∈N}

This is a family of open sets in **R.**

Clearly $\bigcup_{n=1}^{\infty}(-n, n)$ = **R**

∴ {(-n, n)}: n∈N} is an open cover for **R** and this open cover has no finite subcover.

∴ **R** is not compact.

Example 2 : (0,1) with usual metric is not compact.

Proof: Consider the family of open intervals $\{(\frac{1}{n}, n)$: n= 2,…}

Clearly $\bigcup_{n=1}^{\infty}(\frac{1}{n}) = (0,1)$

$\{(\frac{1}{n}, 1)$: n= 2,…} is an open cover for (0,1) and this open cover has a finite subcover.

Hence (0,1) is not compact.

Example 3 : [0,1) with usual metric is not compact.

Proof: Consider the family of open interval {[0, n): n= 1,2, …}

Clearly $\bigcup_{n=1}^{\infty}[0, n) = [0,\infty)$

$\therefore \{[0, n): n = 1,2, \ldots\}$ is an open cover for $[0,\infty)$ and this open cover has no finite subcover.

Hence $[0,\infty)$ is not compact.

Example 4 : Let M be an infinite set with discrete metric. Then M is not compact.

Proof: Let $x \in M$. Since M is discrete metric space $\{x\}$ is open in M.

Also $\bigcup_{x \in M} \{x\} = M$

Hence $\{\{x\}\}$ such that $x \in M$ is an open cover for M and since M is infinite, this open cover has no finite subcover.

Hence M is not compact.

Theorem 5.1 : Let M be a metric space. Let $A \subseteq M$. Then A is compact iff given a family of open sets $\{G_\alpha\}$ in M such that $\bigcup G_\alpha \supseteq A$ there exists a subfamily $\{G_{\alpha 1}, G_{\alpha 2}, \ldots, G_{\alpha n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

Proof: Assume that A be a compact subset of M.

Let $\{G_\alpha\}$ be a family of open sets in M such that $\bigcup G_\alpha \supseteq A$.

To prove: there exist a subfamily $\{G_{\alpha 1}, G_{\alpha 2}, \ldots, G_{\alpha n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

Since $\bigcup G_\alpha \supseteq A$ we get $(\bigcup G_\alpha) \cap A = A$

$(\bigcup G_\alpha \cap A) = A$.

Also, $G_\alpha \cap A$ is open in A.

$\therefore$ The family of $\{G_\alpha \cap A\}$ is open cover for A.

But A is compact. $\therefore$ This open cover has a finite subcover (say) $\{G_{\alpha 1} \cap A, G_{\alpha 2} \cap A, \ldots, G_{\alpha n} \cap A\}$ such that $\bigcup_{i=1}^{n} (G_{\alpha i} \cap A) = A$

$\therefore (\bigcup_{i=1}^{n} (G_{\alpha i}) \cap A = A$

$\therefore \bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$

Conversely assume that a family of opensets $\{G_\alpha\}$ in M such that $\bigcup G_\alpha \supseteq A$ there exist a subfamily.

To Prove: A is compact.

Let $\{H_\alpha\}$ be an open cover for A.

$\therefore$ Each H_α is open in A.

$\therefore H_\alpha = G_\alpha \cap A$ where G_α is open in M.

We have, $\cup H_\alpha = A$

$\bigcup (G_\alpha \cap A) = A$

ie, $\bigcup G_\alpha \cap A = A$

$\therefore \bigcup G_\alpha \supseteq A$

Hence by hypothesis there exist a finite subfamily.

$\{G_{\alpha 1}, G_{\alpha 2}, \ldots, G_{\alpha n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha i}) \cap A = A$

$\therefore \bigcup_{i=1}^{n} (G_{\alpha i} \cap A) = A$

$\therefore \bigcup_{i=1}^{n} H\alpha_i = A$

$\therefore$ There $\{H_{\alpha 1}, H_{\alpha 2}, \ldots, H_{\alpha n}\}$ is a finite subcover of the open cover H_α

$\therefore$ A is compact.

Theorem 5.2 : Any compact subset A of a metric space M is bounded.

Proof: Given, $A \subseteq M$ & A is compact

To Prove: A is bounded

Let $x_0 \in A$

Consider $\{B(x_0, n): n \in N\}$

$\therefore \bigcup_{n=1}^{\alpha} B(x_0, n) \supseteq A$

$\therefore \{B(x_0, n): n \in N\}$ is an open cover for A.

But A is compact

$\therefore$ This open cover has a finite subcover

$\{B(x_0, n_1), B(x_0, n_1), \ldots., B(x_o, n_k)\}$ such that $\bigcup_{i=1}^{k} B(x_0, n_i) \supseteq A$

Let n_o = max $\{n_1, n_2, \ldots, n_k\}$

$\therefore \bigcup_{i=1}^{k} B(x_0, n_i) = B(x_0, n_0)$

Hence $B(x_0, n_0) \supseteq A$

We know that $B(x_0, n_0)$ is bounded set and a subset of a bounded set is bounded.

Hence A is bounded.

Note : The converse of the above theorem is not true.

ie, A bounded set need not be a compact.

For example (0,1) is a bounded subset of R , but (0,1) is not compact.

Theorem 5.3 : Any compact subset A of a metric space (M, d) is closed.

Proof: Given that A is a compact subset of a metric space M.

To Prove: A is closed.

ie, To prove A^c is open

Let $y \in A^c$ & Let $x \in A$

Then $x \neq y$

$\therefore d(x, y) = r_x > 0$

Also, we have $B(x, \frac{rx}{2}) \cap B(y, \frac{rx}{2}) = \emptyset$

Now consider the collection have $\{B(x, \frac{rx}{2}) : x \in A\}$

Clearly $\bigcup_{x \in A} B(x, \frac{rx}{2}) \supseteq A$

Since A is compact, there exist a finite number of such open ball say $B(x_1, \frac{r_{x1}}{2})$, $B(x_2, \frac{r_{x2}}{2})$, …, $B(x_n, \frac{r_x}{2})$ such that $\bigcup_{i=1}^{n} B(x_i, r_{xi/2}) \supseteq A$

Now let $V_y = \bigcap_{i=1}^{n} B(y, r_{xi/2})$

Now let V_y = i=1nB(xi, rx□/2)

Clearly, V_y is an open set containing y.

Since B $(x, \frac{rx}{2}) \cap$ B $(y, \frac{rx}{2}) = \emptyset$ we have $V_y \cap B(x, r_{xi/_2})$ for each i= 1,2, …., n

$\therefore$ $V_y \cap [\bigcup_{i=1}^{n} B(x, r_{xi/_2})] = \emptyset$

$\therefore$ $V_y \cap A = \emptyset$

$\therefore$ $V_y \subseteq A^c$

$\therefore \bigcup_{y \in A^c} V_y = A^c$

Since each V_y is open we have arbitrary union of open set is open

$\therefore \bigcup_{y \in A^c} V_y$ *is open.*

$\therefore$ A^c is open

Hence A is closed.

Note 1 : The converse of the above theorem is not true.

For example $[0,\infty)$ is a closed subset of R but it is not compact.From theorem (5.2) & (5.3) we have any compact subset of a metric space is closed and bounded.

Theorem 5. 4 : A closed subspace of a compact metric space is compact.

Proof: Let M be a compact metric space.

Let A be a non-empty closed subset of M.
We claim that A is compact.
Let $\{G_\alpha : \alpha \in I\}$ be a family of open sets in M such that $\bigcup_{\alpha \in I} G_\alpha \supseteq A$
$A^c \cap [\bigcup_{\alpha \in I} G_\alpha] = M$
Since A is closed A^c is open
$\therefore \{G_\alpha : \alpha \in I\} \cup A^c$ is an open cover for M
Since M is compact, this open cover has a finite subcover $G_{\alpha 1}, G_{\alpha 2}, \ldots, G_{\alpha n}$, A^c such that
$\bigcup_{i=1}^{n} G_{\alpha i} \cup A^c = M$
$\therefore \bigcup_{i=1}^{n} G_{\alpha i} \supseteq A$
$\therefore$ A is compact.

Compact Subsets of R

Theorem 5.5 (Heine Borel Theorem): Any closed interval [a, b] is a compact subset of **R.**

Proof: Let $\{G_\alpha : \alpha \in I\}$ be a family of open sets in M such that $\bigcup_{\alpha \in I} G_\alpha \supseteq [a, b]$

We claim that [a, b] is a compact subset of R.

Let S = $\{x: x \in [a, b]\}$ and $[a, x]$ can be covered by finite number of $G_{\alpha's}$.

Clearly $a \in S$ and hence $S \neq \emptyset$.

Also, S is bounded above by b.

Let 'c' denote the least upper bound of S.

Clearly, $c \in [a, b]$

$\therefore c \in G_{\alpha 1}$ for some α_1

Since $G_{\alpha 1}$ is open, there exist $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \subseteq G_{\alpha 1}$

Choose $x_1 \in [a, b]$ such that $x_1 < c$ and $[x_1, c] \subseteq G_{\alpha 1}$

Now, since $x_1 < c$, $[a, x_1]$ can be covered by finite number of $G_{\alpha's}$.

These finite number of $G_{\alpha's}$ together with $G_{\alpha 1}$ covers [a, c]

$\therefore$ By definition of S, $c \in S$.

Now we claim that c=b

Suppose $c \neq b$. Then choose $x_2 \in [a, b]$ such that $x_2 > c$ and $[c, x_2] \subseteq G_{\alpha 1}$

As before $[a, x_2]$ can be covered by finite number of $G_{\alpha's}$.

Hence $x_2 \in S$

But $x_2 > c$ which is a contradiction since c is the lub of S.

$\therefore c = b$

$\therefore$ [a, b] can be covered by finite number of $G_{\alpha's}$.

$\therefore$ [a, b] is a compact subset of R.

Theorem 5.6 : A subset A of R is a compact iff A is closed and bounded.

Proof: Assume that A is compact.

To prove: A is closed and bounded.

By theorem 5.2, We have A is bounded

By theorem 5.3, We have A is closed

Conversely, assume that a subset of R which is closed and bounded.

To prove: A is compact

Let A be a subset of R.

Since A is bounded we can find a closed interval [a, b] such that $A \subseteq [a, b]$

Since A is closed in R, A is closed in [a, b] also.

Thus, A is closed subset of a compact metric space [a, b].

Hence by theorem 5.4, A is compact.

EQUIVALENT CHARACTERISATION FOR COMPACTNESS

Definition : A family τ of subset of a set M is said to have the finite intersection property if any finite subfamily of τ has non-empty intersection.

Example : In R the family of closed intervals $\tau = \{[-n, n]: n \in N\}$ has finite intersection property.

Theorem 5.7 : A metric space M is compact iff any family of closed sets with finite intersection property has non-empty intersection.

Proof: Assume that M is compact.

Let $\{A_\alpha\}$ be a family of closed subsets of M with finite intersection property.

We claim that $\cap A_\alpha \neq \emptyset$

Suppose $\cap A_\alpha = \emptyset$

Then $(\cap A\alpha)^c = \emptyset^c$

$\cup A_\alpha^c = M$

Also, since each A_α is closed A_α^c is open

$\therefore \{A_\alpha^c\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say $A_1^c, A_2^c, \ldots, A_n^c$

$\therefore \cup_{i=1}^{n} A_i^c = M$

$\therefore (\cap_{i=1}^{n} A_i)^c = M$

$[(\cap_{i=1}^{n} A_i)^c]^c = M^c$

$\cap_{i=1}^{n} A_i = \emptyset$

which is the contradiction to the finite intersection property

$\therefore \cap A_\alpha \neq \emptyset$

Conversely assume that each family of closed sets in M with finite intersection property has non-empty intersection.

To prove M is compact

Let G_α such that $G_\alpha \in I$ be an open cover for M

$\therefore \bigcup_{\alpha \in I} G_\alpha = M$

$\therefore (\bigcup_{\alpha \in I} G_\alpha)^c = M^c$

$\therefore \bigcap_{\alpha \in I} G_\alpha{}^c = \emptyset$

Since G_α is open G_α^c is closed for each α.

$\therefore \tau = \{G_\alpha^c : \alpha \in I\}$ is a family of closed sets whose intersection is empty.

Hence by hypothesis, this family of closed sets does not have the finite intersection property.

Hence there exist a finite subcollection of τ say $\{G_1^c, G_2^c, \ldots, G_n^c\}$ such that

$\bigcap_{i=1}^{n} G_i = \emptyset$

$ie., (\bigcup_{i=1}^{n} G_i)^c = \emptyset$

ie., $\bigcup_{i=1}^{n} G_i = M$

$\therefore \{G_1, G_2, \ldots, G_n\}$ is a finite subcover of the given open cover.

Hence M is compact.

Definition : A metric space M is said to be totally bounded if for every $\varepsilon > 0$ there exist a finite number of elements $x_1, x_2, \ldots, x_n \in M$ such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup, \ldots, \cup B(x_n, \varepsilon) = M$.

A non- empty subset A of a metric space M is said to be totally bounded if the subspace A is a totally bounded metric space.

Theorem 5.8 : Any compact metric space is totally bounded.

Proof: Let M be a compact metric space.

Then $\{B(x, \varepsilon) : x \in M\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say $B(x_1, \varepsilon), B(x_2, \varepsilon), \ldots,$

$B(x_n, \varepsilon)$ such that $\bigcup_{i=n}^{n} B(x_i, \varepsilon) = M$

ie., $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots\ldots \cup B(x_n, \varepsilon) = M$.

Hence M is totally bounded.

Theorem 5.9 : Let A be a subset of a metric space M. If A is totally bounded, then A is bounded.

Proof: Let A be a totally bounded subset of M.

Let ε>o be given, then there exist a finite number of points $x_1, x_2, \dots, x_n \in A$ such that B (x_1, ε) ∪ B (x_2, ε) ∪........... ∪ B (x_n, ε) = A where B (xi,ε) is an open ball in A.

Further we know that an open ball is a bounded set.

Thus, A is the union of a finite number of bounded sets and hence A is bounded.

Note : The converse of the above theorem is not true. ie., a bounded set need not be totally bounded.

Example : Let M be an infinite set with discrete metric.

Clearly M is bounded

Now B(x ,$\frac{1}{2}$) = $\{x\}$

Since M is infinite M cannot be written as the union of a finite number of open balls B(x ,$\frac{1}{2}$).

Then M is not totally bounded.

Definition : Let x_n be a sequence in a metric space M. Let $n_1 < n_2 < \dots < n_k < \dots$ be an increasing sequence of positive integers. Then (x_{nk}) is called the subsequence of (x_n)

Theorem 5.10 : A metric space (M, d) is totally bounded iff every sequence in M has the Cauchy's subsequence.

Proof: Suppose every sequence in M has a Cauchy subsequence.

We claim that M is totally bounded

Let ε> 0 be given

Choose $x_1 \in$M

If B (x_1, ε)$\neq M$, choose $x_2 \in$M-B (x_1, ε) so that D (x_1,x_2)$\geq \varepsilon$

Now B (x_1, ε)∪ B (x_2, ε) = M then the proof is complete.

If not choose $x_3 \in$M- [B (x_1, ε)∪ B (x_2, ε)] and so on.

Suppose this process does not stop at a finite stage.

Then we obtain a sequence$x_1, x_2, \dots, x_n$, ... such that d (x_n, x_m)$\geq \varepsilon$ if n$\neq m$

Clearly this sequence (x_n) cannot have a Cauchy subsequence which is a contradiction.

Hence the above process stops at a finite stage, and we get a finite set of points $x_1, x_2, \ldots, x_n$, such that M = B $(x_1, \varepsilon) \cup$ B $(x_2, \varepsilon) \cup, \ldots, \cup$ B (x_n, ε)

$\therefore$ M is totally bounded.

Conversely suppose M is totally bounded.

We claim that every sequence in M has a Cauchy subsequence.

Let $S_1 = \{x_{11}, x_{12}, x_{13}, \ldots x_{1n}\}$ be a sequence in M.

If one term of the sequence is infinitely repeated, then S_1 contains a constant subsequence which is obviously a Cauchy subsequence.

Hence, we assume that no term of S_1 is infinitely repeated so that the range S is infinite.

Now, Since M is totally bounded M can be covered by a finite number of open balls of radius $\frac{1}{2}$

Hence at least one of these balls must contain an infinite number of terms of the sequence S_1.

$\therefore$ S_1 contains a subsequence $S_2 = (x_{21}, x_{22}, x_{23}, \ldots x_{2n}, \ldots)$ all terms of which lie within an open ball of radius $\frac{1}{2}$

Similarly, S_2 contains a subsequence $S_3 = \{x_{31}, x_{32}, x_{33}, \ldots x_{3n}, \ldots)$ all terms of which lie within an open ball of radius $\frac{1}{3}$

We repeat this process of forming successive subsequence and finally we take the diagonal sequence.

$S = (x_{11}, x_{22}, x_{33}, \ldots x_{nn}, \ldots)$

We claim that S is a Cauchy subsequence of S_1

If m>n both x_{mm} and x_{nn} lies with an open ball of radius $\frac{1}{n}$

$\therefore$ d $(x_{mm}, x_{nn}) < \frac{2}{n}$

Hence d $(x_{mm}, x_{nn}) < \varepsilon$ if n, m $> \frac{2}{\varepsilon}$

This shows that S is a Cauchy subsequence of S_1

Thus, every subsequence in M contains a Cauchy subsequence.

Corollary : A non-empty subset of totally bounded set is totally bounded.

Proof: Let A be totally bounded subset of a metric space M.

Let B be a non-empty subset of A.

To prove: B is totally bounded.

It is enough to prove that every sequence has a Cauchy subsequence

Let (x_n) be a sequence in B.

$\therefore (x_n)$ is a sequence in A.

Since A is totally bounded (x_n) has a Cauchy subsequence.

Thus, every sequence in B has a Cauchy subsequence.

$\therefore$ B is totally bounded.

Definition : A metric space M is said to be sequentially compact if every sequence in M has a convergent sub-sequence.

Theorem 5.11 : Let be a Cauchy sequence in a metric space. If (x_n) has a subsequence (x_{nk}) converging to$x$, then converges to x.

Proof: Let $\varepsilon > 0$ be given

Since (x_n) is a Cauchy sequence, there exists a positive integer m_1 such that $d(x_n, x_m) < \frac{1}{2}\varepsilon$ for all $n, m \geq m_1$ …… (1)

Also, since $(x_{nk}) \to x$, there exists a positive integer m_2 such that

$d(x_{nk}, x) < \frac{1}{2}\varepsilon$ for all $n_k \geq m_2$ …. (2)

Let $m_0 = \max\{m_1, m_2\}$ and fix $n_k \geq m_0$

Then $d(x_n, x) \leq d(x_n, x_{nk}) + d(x_{nk}, x_n)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ for all $n_k \geq m_0$

$= \varepsilon$ for all $n_k \geq m_0$ $\therefore (x_n) \to x$

Theorem 5. 12 : In a metric space M, the following are equivalent.

(i) M is compact.

(ii) Any infinite subset of M has a limit point.

(iii) M is sequentially compact.

(iv) M is totally bounded and complete.

Proof: (i)⇒ (ii).

Assume that M is compact

To prove: Any finite subset of M has a limit point.

Let A be an infinite subset of M

Suppose A has no limit point in M

Let, $x \in$M

Since x is not a limit point of A there exists an open ball B (x,r_x) such that

B (x,r_x)∩ (A- {x}) = ∅

$$B\ (x, r_x) \cap A = \begin{cases} \{x\}, if\ x \in A \\ \emptyset\ ,\ if\ x \notin A \end{cases}$$

Now {B (x,r):$x \in$M} is an open cover for M

Also, each B (x,r_x) cover atmost one point of the finite set A.

Hence this open cover cannot have a finite subcover which is a contradiction to (i)

Hence A has atleast one limit point.

(ii)⇒(iii)

Assume that: Any finite subset of M has a limit point.

To prove: M is sequentially compact.

Let (x_n) be a sequence in M.

If one term of the sequence is infinitely repeated, then (x_n) contains a constant subsequence which is convergent.

Otherwise (x_n) has an infinite number of terms.

By hypothesis, this infinite set has a limit point, say x.

We know that "for any r>0, the open ball B (x,r) contains infinite number of terms of the sequence (x_n)."

Now we choose a positive integer n_1, such that $x_{n1} \in B(x, 1)$

Then choose $n_2 > n_1$ such that $x_{n2} \in B(x, \frac{1}{2})$

In general, for each positive integer k choose n_k such that $n_k > n_{k1}$ and $x_{nk} \in B(x, \frac{1}{k})$

Clearly, (x_{nk}) is a subsequence of (x_n).

Also, $d(x_{nk}, x) < \frac{1}{k}$

$\therefore (x_{nk}) \to x$

Thus (x_{nk}) is a convergent subsequence of (x_n).

Hence M is sequentially compact.

(iii)⇒(iv)

Assume that M is sequentially compact.

To prove: M is totally bounded and complete.

By hypothesis, every sequence in M has a convergent subsequence.

But every convergent sequence is a Cauchy sequence.

Thus, every sequence in M has a Cauchy sequence.

By theorem, M is totally bounded.

Now, we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

By hypothesis, (x_n) contains a convergent subsequence in (x_{nk})

Let $(x_{nk}) \to x$ (say)

Now by theorem $(x_n) \to x$

$\therefore$ M is complete.

(iv) ⇒(i)

Assume that M is totally bounded and complete.

To prove: M is compact.

Suppose M is not compact.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover.

Let $r_n = \frac{1}{2^n}$

Since, M is totally bounded, M can be covered by a finite number of open balls of radius r_1.

Since M cannot be covered by a finite number of G_{α}'s atleast one of these open balls, say

$B(x_1, r_1)$ cannot be covered by a finite number of G_{α}'s.

Now $B(x_1, r_1)$ is totally bounded.

Hence as before we can find $x_2 \in B(x_1, r_1)$ such that $B(x_2, r_2)$ cannot be covered by a finite number of G_{α}'s.

Proceeding like this we obtain a sequence (x_n) in M such that $B(x_n, r_n)$ cannot be covered by a finite number of G_{α}'s and $x_{n+1} \in B(x_n, r_n)$ for all n.

Now, $d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p})$

$< r_n + r_{n+1} + \dots + r_{n+p+1}$

$$= \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^{n+p-1}}$$

$$= \frac{1}{2^{n-1}}\left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^p}\right)$$

$< \frac{1}{2^{n-1}}$

$\therefore (x_n)$ is a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that $B(x, \varepsilon) \subseteq G_\alpha$ …… (1)

We have $(x_n) \rightarrow x$ and $(r_n) = \left(\frac{1}{2^n}\right) \rightarrow 0$

Hence, we find a positive integer n_1 such that $d(x_n, x) < \frac{1}{2}\varepsilon$ and $r_n < \frac{1}{2}\varepsilon$ for all $n \geq n_1$

We claim that B $(x_n, r_n) \subseteq$ B (x, ε)

Let $y \in$ B (x_n, r_n)

$\therefore$ d $(y, x_n) < r_n < \frac{1}{2}\varepsilon$

Now, d $(y, x) \leq$ d (y, x_n) +d (x_n, x)

$\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon$

$= \varepsilon$

$\therefore y \in$ B (x, ε)

$\therefore$ B $(x_n, r_n) \subseteq$ B $(x, \varepsilon) \subseteq G_\alpha$ [by (1)]

Thus B (x_n, r_n) is covered by the single set G_α which is a contradiction.

Since B (x, r_n) cannot be covered by a finite number of G_{α}'s.

Hence M is compact.

Theorem 5.13 : R with usual metric is complete.

Proof: Let (x_n) be a Cauchy sequence in R.

Then (x_n) is a bounded sequence and hence is contained in a closed interval [a, b].

Now [a, b] is compact and hence is complete.

Hence (x_n) converges to some point $x \in$[a, b]

Thus, every Cauchy sequence (x_n) in R converges to some point x in R and hence R is complete.

Solved Problems

Problem 1: Given an example of a closed and bounded subset of l_2 which is not compact.

Solution:

Consider 0 = (0,0,0, …)$\in l_2$

Consider the closed ball B [0,1]

Clearly, B [0,1] is bounded.

Also, B [0,1] is closed set.

We claim that B [0,1] is not compact.

Consider $e_1 = (1,0,0, \ldots)$.

$e_2 = (0,1, 0\ldots)$; $e_n = (0, 0, 0\ldots1, 0 \ldots)$

Now, d (0, e_n) = 1 and hence $e_n \in$B [0,1] for all n

Thus (e_n) is a sequence in B [0,1]

Also, d (e_n, e_m) = $\sqrt{2}$ if n $\neq m$.

Hence the sequence (e_n) does not contain a Cauchy subsequence.

$\therefore$ B [0,1] is not totally bounded.

$\therefore$ B [0,1] is not compact.

Problem 2 : Prove that any totally bounded metric space is separable.

Solution: Let M be a totally bounded metric space.

For each natural number n.

Let $A_n = \{ x_{n1}, x_{n2}, \ldots x_{nn}\}$ be a subset of M such that $\bigcup_{i=1}^{k} B(x_{ni}, \frac{1}{n})$ = M …. (1)

Let A = $\bigcup_{n=1}^{\infty} A_n$

Since each A_n is finite, A is countable subset of M.

We claim that A is dense in M.

Let B (x,ε) be any open ball.

Choose a natural number n such that $\frac{1}{n}<\varepsilon$

Now, $x \in$B (x_{ni}, $\frac{1}{n}$) for some i [by (1)]

$\therefore$ d (x_{ni}, x) $<\frac{1}{n}<\varepsilon$

$\therefore$ (x_{ni},) $\in$ B (x,ε)

$\therefore$ B (x,ε) $\cap A \neq \emptyset$

Thus, every open ball in M has non-empty intersection with A.

Hence by theorem, A is dense in M.

Thus, A is a countable dense subset of M.

Hence M is separable.

Problem 3 : Prove that any bounded sequence in R has a convergent subsequence.

Solution: Let (x_n) be a bounded sequence in R.

Then there exists a closed interval [a, b] such that $x_n \in [a, b]$ for all n.

Thus (x_n) is sequence in the compact metric space [a, b]

Hence by theorem, (x_n) has a convergent subsequence.

Problem 4 : Prove that the closure of a totally bounded set is totally bounded.

Solution: Let. A be a totally bounded Subset of a metric space M.

We claim that $\overline{A}$ is totally bounded.

We shall show that every sequence in A contains a Cauchy subsequence.

Let (x_n) be a sequence in $\overline{A}$.

Let ε>0 be given

Then since the $x_n \in \overline{A}$ $B(x_n, \frac{1}{3}\varepsilon) \cap A \neq \emptyset$

Choose $y_n \in B(x_n, \frac{1}{3}\varepsilon) \cap A$

$\therefore d(y_n, x_n) < \frac{1}{3}\varepsilon$ …. (1)

Now, (y_n) is totally bounded (y_n) contains a Cauchy subsequence say (y_{nk}).

Hence there exists a natural number m such that

$d(y_{ni}, x_{nj}) < \frac{1}{3}\varepsilon$ for all $n_i, n_j \geq m$ …. (2)

$\therefore$ d (x_{ni}, y_{nj}) < d (x_{ni}, y_n) +d (y_{ni}, y_{nj}) +d (y_{nj}, x_{nj})

$<\frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon$ for all n_i, n_j [by (1) and (2)]

Hence (x_{nk}) is a Cauchy subsequence of x_n.

$\therefore \overline{A}$ is totally bounded.

Problem 5 : Lot A be a totally bounded subset of R. Prove that$\overline{A}$ is compact.

Solution: Since A is totally bounded $\overline{A}$ is also totally bounded.

Also, since$\overline{A}$ is a closed subset of Rand R is complete $\overline{A}$ .is complete.

Hence $\overline{A}$ is to totally bounded and complete.

$\therefore \overline{A}$ is compact.

COMPACTNESS AND CONTINUITY

Theorem 5.14 : Let f be a continuous mapping from a compact metric space $M_1$to any metric spaceM_2. Then f (M_1) is compact.

ie, continuous image of a compact metric space is compact.

Proof: Without loss of generality, we assume that f (M_1) = M_2

Let {G_α} be a family open set in M_2 such that $\cup G_\alpha = M_2$

$\therefore \cup G_\alpha$ = f (M_1)

$\therefore f^{-1} \cup G_\alpha = M_1$

$\therefore \cup f^{-1} (G_\alpha) = M_1$

Also, since f is continuous $f^{-1} (G_\alpha)$ is open in M_1 for each α

$\therefore \{f^{-1} (G_\alpha)\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover say $f^{-1} (G_{\alpha 1})$, … $f^{-1} (G_{\alpha n})$

$\therefore f^{-1} (G_{\alpha 1}) \cup f^{-1} (G_{\alpha 2}) \cup \ldots. \cup f^{-1} (G_{\alpha n}) = M_1$

$\therefore f^{-1}(\cup_{i=1}^{n} G_{\alpha i}) = M_1$

$\therefore \cup_{i=1}^{n} G_{\alpha i}$ = f (M_1) = M_2

$G_{\alpha 1}, G_{\alpha 2}, \ldots G_{\alpha n}$ is an open cover for M_2

Thus, the given open cover $\{G_\alpha\}$ for M_2 has a finite subcover.

$\therefore$ M_2 is compact.

Corollary 1 : Let f be a continuous map from a compact metric space M_1 into any metric M_2. Then f (M_1) is closed and bounded.

Proof: f (M_1) is compact and hence is closed and bounded.

Corollary 2 : Any continuous real valued function f defined on a compact metric space is bounded and attains its bounds.

Proof: Let M be a compact metric space.

Let f: M $\to$R be a continuous real valued function.

Then f (M) be a compact subset of R.

$\therefore$ f (M) is closed and bounded subset of R

Since f (M) is bounded f is a bounded function.

Now, let a = lub of f (M) &

b = glb of f (M)

By definition of lub & glb a, b $\in \overline{f\ (M)}$ but f (M) is closed.

Hence f (M) = $\overline{f\ (M)}$

$\therefore$ a, b $\in$ f (M)

$\therefore$ There exists $x, y \in$M such that f (x) = a & f (y) = b

Hence f attains its bounds.

Note : Corollary 2 is not true if M is not compact.

The function f: (0, 1)$\to$R defined by f (x) = $\frac{1}{x}$ is continuous but not bounded.

Theorem 5.15 : Any continuous mapping f defined on a compact metric space (M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof: Let ε>0 be given

Let $x \in M_1$

Since f is continuous at x,there exist $\delta_x>0$ such that $d_1(y, x) < \delta_x$

$\Rightarrow d_2((f(y), f(x)) < \frac{\varepsilon}{2}$ …. (1)

$\{B(x, \frac{\delta_x}{2}): x \in M_1\}$ is an open cover for M_1.

Since M_1 is compact this open cover has a finite subcover say $B(x, \frac{\delta_{x1}}{2}), \ldots, B(x, \frac{\delta_{xn}}{2})$

Let $\delta = \min\{\frac{\delta_{x1}}{2}, \frac{\delta_{x2}}{2}, \ldots, \frac{\delta_{xn}}{2}\}$

We claim that $d_1(p, q) < \delta \Rightarrow d_2((f(p), f(q)) < \varepsilon$

Let $p \in B(x_i, \frac{\delta_{xi}}{2})$ for some $1 \leq i \leq n$

$\therefore d_1(p, x_i) < \frac{\delta_{xi}}{2}$

$\therefore d_2(f(p), f(x_i) < \frac{\varepsilon}{2}$ …. (2) [from (1)]

Now, $d_1(q, x_i) \leq d_1(p, q) + d_1(p, x_i)$

$< \delta + \frac{\delta_{xi}}{2}$

$< \frac{\delta_{xi}}{2} + \frac{\delta_{xi}}{2}$

$= \delta_{xi}$

Thus $d_1(q, x_i) < \delta_{xi}$

$\therefore d_2(f(q), f(x_i) < \frac{\varepsilon}{2}$ …. (3) [from (1)]

Now, $d_2((f(p), f(q)) \leq d_2(f(p), f(x_i) + d_2(f(x_i), f(q))$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

Thus, $d_1(p, q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon$

$\therefore$ f is uniformly continuous on M_1.

Theorem 5.16 : Let f be a 1-1 continuous function from a compact metric space M_1 onto any metric space M_2. Then$f^{-1}$is continuous on M_2. Hence f is a homeomorphism from M_1 onto M_2.

Proof: We shall show that $f^{-1}$is continuous.

By proving that F is closed set in $M_1 \Rightarrow (f^{-1)-1}(F) = f(F)$ is a closed set in M_2.

Let F be a closed set in M_1

Since$M_1$is compact

$\therefore$ F is compact

Since f is continuous, f (F) is a compact subset of M_2.

$\therefore$ f (F) is closed subset of M_2.

$\therefore f^{-1}$ is continuous on M_2.

Solved Problems

Problem 1: Prove that the range of a continuous real valued function f on a compact connected metric space M must be either a single point or a closed and bounded interval.

Solution: Let $f:M \rightarrow R$ be a continuous function.

If f is a constant function, then the range of f is a single point.

Suppose f is not a constant function, then the range of f contains more than one point.

Since M is connected

f (M) is connected subset of R

$\therefore$ f(M) is an interval in R.

Also, since Mis compact and fis continuous.

$\therefore$ f (M) is a compact subset of R.

$\therefore$ f (M) is a closed and bounded subset of R.

Thus f (M) is a closed and bounded interval of R.

Problem 2 : Prove that any continuous function f: [a,b] R is not onto.

Solution: Suppose f is onto

Then f [a, b] =R

Now, since [a, b] is compact and fis continuous.

$\therefore$ f [a,b] = R is compact, whichis a contradiction.

$\therefore$ f is not onto.

EXERCISES

1. Give an example of an open cover which has no finite subcover for the following subsets of R.

   (i) (5,6) (ii) $(5,\infty)$ (iii) $[5,\infty)$ (iv) [7,9].
2. Show that every finite metric space is compact.
3. Give an example of a connected subset of R which is not compact.
4. If A and B are two compact subsets of a metric space M, prove that AUB is also compact.
5. Determine which of the following subsets of R are compact.

   (i) Z (ii) Q (iii) [1,2] (iv) (3,4)

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